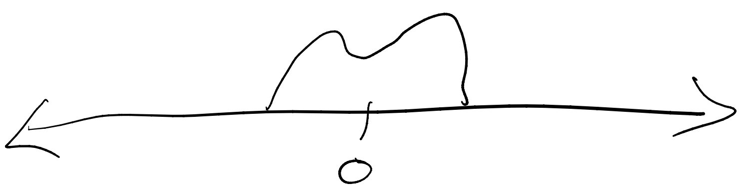
Tuesday, September 15, 2020 6:01 PM

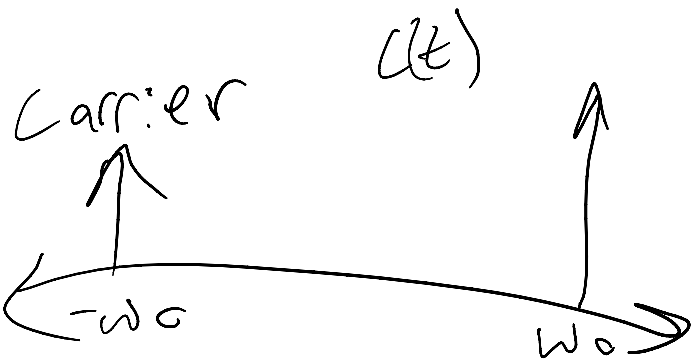


in Matlab

: $m[n]$ sampled at $f_s^{(0)}$

~8 kHz

~40 kHz



$f_c > 100$ kHz

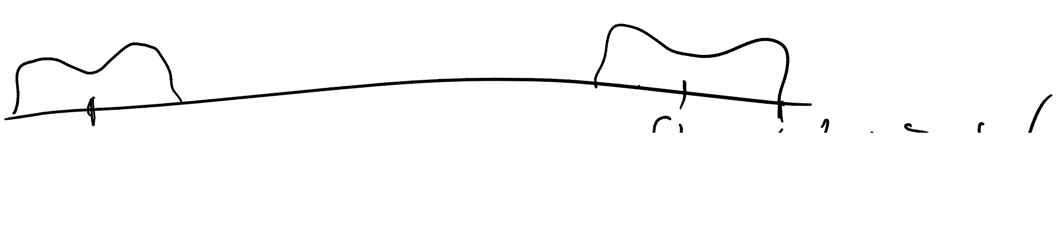
$c[k]$ sampled at $f_s^{(1)}$

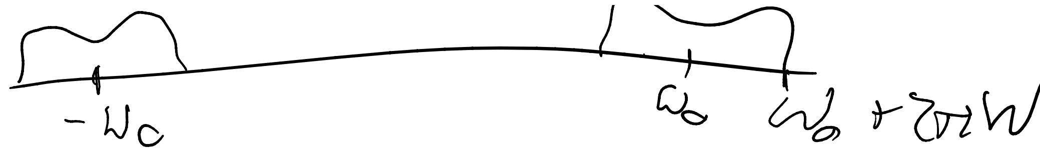
$f_s^{(1)}$ to represent $c(t)$ must be $\geq 2f_0$ ← Nyquist sampling theorem

so to take $m(t)c(t) \rightarrow$ Matlab

$m[k]c[k]$ at $f_s^{(1)}$

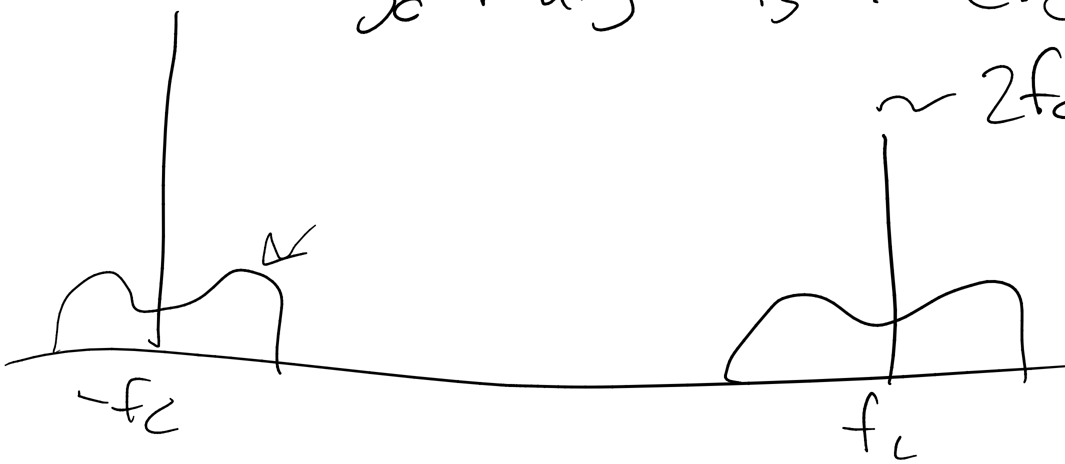
and





So really $f_s^{(1)} > 2f_c + 2W$

$\sim 2f_c + 40\,\text{kHz}$



mag plots - semilogy

phase - unwrap

Prob Review

Tuesday, September 15, 2020 6:08 PM

Samplespace Ω — set of outcomes of a random exp.

Prob. fnch is $P: \mathcal{P}(\Omega) \to \mathbb{R}$ satisfying

1) $0 \leq P(E) \leq 1 \quad \forall E \subset \Omega$

2) $P(\Omega) = 1$

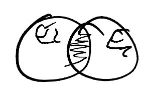
3) if $E_1, \dots, E_k$ are mutually exclusive ($E_i \cap E_j = \emptyset \ \forall i,j$) then $P(\bigcup_i E_i) = \sum_i P(E_i)$

if a samplespace is countable — discrete

uncountable — cts

$P(E^c) = 1 - P(E)$

$P(\emptyset) = 0$

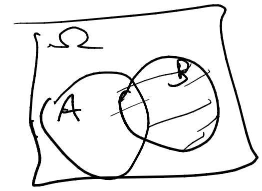


$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

if $E_1 \subset E_2$, $P(E_1) \leq P(E_2)$

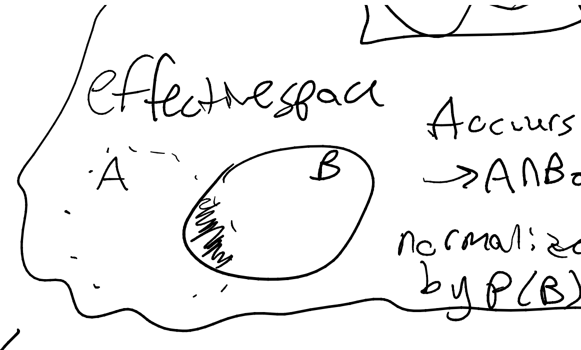
Def A, B <u>indepenents</u> iff $P(A \cap B) = P(A)P(B)$

Def $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$



effective space

and $P(X|B)$ satisfies the def of a prob.



A occurs $|B$ $\to A \cap B$ occurs

normalized dist. by $P(B)$

fncn in variable X w/ sample space Ω if $P(B) \neq 0$

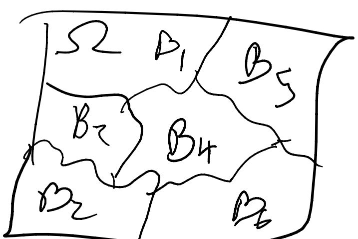
If events are indep, $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$

$$P(A|B) = P(A)$$

<u>Def.</u> A <u>partition</u> of the sample space Ω is a set of events $B_1, \ldots, B_n$ satisfying

1) $\bigcup_{i=1}^{n} B_i = \Omega$ (exhaustive)

2) $B_i \cap B_j = \emptyset \ \forall i,j$ (mutual exclusivity)



<u>Total Probability Theorem</u>: If $\{B_i\}_{i=1}^{n}$ is a partition of Ω, then for any $A \subset \Omega$, I can write

$$\boxed{P(A) = \sum_{i=1}^{n} P(B_i)\, P(A|B_i)}$$

<u>Proof.</u> $P(B_i)P(A|B_i) = P(A\cap B_i) \ \forall i$

so $\sum_i P(B_i) P(A \mid B_i) = \sum_i P(A \cap B_i)$

if B_i's are m.e., then $A \cap B_i$'s are m.e.

so by Prob. def (3),

$$= P\left(\bigcup_i (A \cap B_i)\right)$$

but $\bigcup_i B_i = \Omega$, so $= P(A)$ //

Bayes' Theorem $P(B \mid A) = \dfrac{P(A \mid B) P(B)}{P(A)}$

Proof $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$, $P(B \cap A) = P(A \mid B) P(B)$

$$P(B \mid A) = \frac{P(A \mid B) P(B)}{P(A)} \quad //$$

an alternate (equivalent) statement:
if $\{B_i\}_{i=1}^n$ is a partition

$$\boxed{P(B_k \mid A) = \frac{P(A \mid B_k) P(B_k)}{\sum_i P(A \mid B_i) P(B_i)}}$$

$P(B_i)$ "priors"

$P(B_i)$ "priors"

$P(B_k|A)$ "posterior"

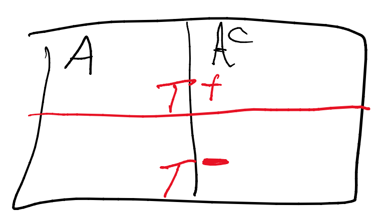
Ex. Covid19 antibody test has false neg. rate of 4%, false pos. rate of 10%. You take the test and you test positive. What is the prob. that you have the antibodies given 20% of essential workers like you have them?

A = having antibodies, B = Testing positive

(partition: $\{A, A^c\}$)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)}$$

$$= \frac{(1-P(B^c|A))(.2)}{(1-P(B^c|A))(.2)+(.1)(.8)}, \quad P(B^c|A) = .04$$



$$= \frac{(.96)(.2)}{(.96)(.2)+.08} \approx 71\%$$

<u>Random Variables</u> are functions from sample space to $\mathbb{C}$

$X : \Omega \to \mathbb{C}$ (or $\mathbb{R}$)

well, Ω may not be numerical

well, Ω may not be numerical

$\Omega = \{\text{Heads}, \text{Tails}\}$

$X(H) = 0$ $\qquad$ $Y(H) = -1$

$X(T) = 1$ $\qquad$ $Y(T) = 1$

Same exp., 2 vars, 2 different means

use r.vs as a means of mapping Ω to a space where we can "do math" and define things like distributions or Expectations

we call range(X) "the space" of X, S.

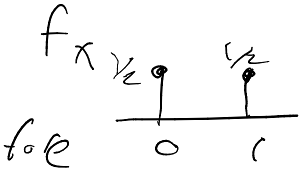
if S countable, then X is <u>discrete</u>

uncountable, then X is <u>cts</u>

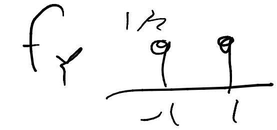
<u>Def.</u> P.M.F. for a r.v. X with space S as

$$f_X : S \to [0, 1]$$

from before



$$f_X(x_i) = P(X = x_i)$$



$$:= P(X^{-1}(x_i))$$

subset of Ω

$\{\omega \mid X(\omega) = x_i\} \subset \Omega$

$\sum f_X(x_i) = 1$, and def. $f_X(x) = 0$ if $x \notin S$

$\sum_{x_i \in S} f_X(x_i) = 1$, and def. $f_X(x) = 0$ if $x \notin S$

cts case: P.D.F. of cts r.v. X space S

$f: S \to \mathbb{R}^+$, $\int_I f_X(y)\,dy = P(X \in I)$

$$:= P(X^{-1}(I))$$

$$\int_{\mathbb{R}} f_X(y)\,dy = 1$$

CDF:

$$F_X(y) = \int_{-\infty}^{y} f_X(z)\,dz = P(X \leq y)$$

(can be def. for discrete r.v.s)

a) $0 \leq F_X(x) \leq 1$, b) F_X is nondecreasing

c) $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$

d) $P(X \in (a,b)) = F_X(b) - F_X(a)$

e) $f_X(x) = \frac{d}{dy} F_X(y) \Big|_{y=x}$

$$f_X(x) = \frac{d}{dy} F_X(y) \Big|_{y=x}$$

Some important r.v.s

1. **Bernoulli** experiment is a "success" with prob. p
"failure" with prob. $1-p$

good model for some binary channels, bitwise

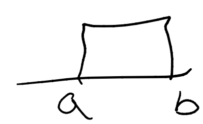
2. **Binomial** I perform n Bernoulli trials (indep), counts # successes

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

"how many bit errors in an n-bit communication"

3. Uniform $X = U(a,b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{else} \end{cases}$$



4. Normal (Gaussian) $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

common model for thermal noise
(most types of noise)

Q function: $Q(x) = \int^{\infty} f(x)\,dx$ where

Q function: $Q(x) = \int_x f_X(x)\,dx$ where $X = N(0,1)$

in Matlab, its inverse is too, $Q(0) = 1/2$

if $X = N(0,1)$, then

not squared!

$$\boxed{N(\mu, \sigma^2) = \sigma X + \mu}$$

generate w/ randn

multiply by

Expectation

disc: $E[X] = \sum_i x_i f_X(x_i)$

cts: $E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$E[X] = \mu_X$, $E[(X-\mu)^2] = \sigma_X^2$ ← variance

mean

σ_X "std. deviation"

expectation is linear!

$E[aX+b] = aE[X] + b$

if $Y = g(X)$, then $E[g(X)] = \sum g(x) f_X(x)$

if $Y = g(X)$, then $E[g(X)] = \sum g(x) f_X(x)$

or

$\int g(x) f_X(x)\,dx$

$N(\mu, \sigma^2)$ has mean μ, var. σ^2

Multiple R.V.

joint pmf. $f_{X,Y}(x,y) = P(X=x \text{ and } Y=y)$

marginal pmf $f_X(x) = \sum_{y \in S_y} f_{X,Y}(x,y)$

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)\,dy$$

joint CDF

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv = P(X \le x \text{ and } Y \le y)$$

two random variables are indep.

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Conditional dist

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & f_Y(y) \neq 0 \\ 0, & \text{else} \end{cases}$$

$$x|y \quad \begin{cases} \ldots \\ 0, & \text{else} \end{cases}$$

if X, Y indep. then $f_{X|Y}(x|y) = f_X(x)$

Correlation b/t X and Y: $r_{X,Y} = E[XY]$

Covariance: $\sigma_{X,Y} = E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[XY] - E[X]E[Y]$$

Correlation Coefficient: $\rho_{X,Y} = \dfrac{\sigma_{X,Y}}{\sigma_X \sigma_Y}$

We say X, Y are orthogonal if $E[XY] = 0$

that is — if $r_{X,Y} = 0$

we say X, Y are uncorrelated if $\sigma_{X,Y} = 0 \Rightarrow \rho_{X,Y} = 0$

if X, Y indep: $\sigma_{X,Y} = E[XY] - E[X]E[Y]$

$$= E[X]E[Y] - E[X]E[Y] = 0$$

$\boxed{\text{indep} \Rightarrow \text{uncorrelated}}$

converse is false

Central limit Theorem

If $X_1, \ldots, X_n$ are indep., identically distributed

$$\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow N(\mu, \sigma^2/n) \text{ as } n \nearrow$$

where μ, σ^2 are mean, var of X_i

Random Process

Tuesday, September 15, 2020 7:37 PM

A random process is a generalization of the r.v.

It is a set of possible functions of time rather than values

Ex.

suppose $X(t) = A\cos(w_0 t + \Theta)$, $\Theta = U(0, 2\pi)$

↑ random process

at any point t_0 in time, $X(t_0)$ is a random variable

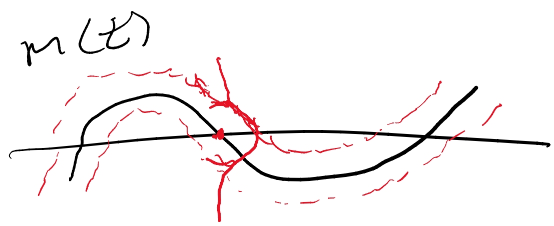
$A\cos(w_0 t_0 + \Theta)$

A random process is a function of a real variable t (time) which is a random variable at each value of t.

Ex.

$X(t) = m(t) + N(0, \sigma^2)$

$= N(m(t), \sigma^2)$



Discrete time random process = at each instance in discrete time is a r.v. = $\{X_i\}^{\infty}$ where each X_i is an r.v is a r.v. $\equiv \{X_n\}_{n=-\infty}^{\infty}$ where each X_n is an r.v.

Expectation

Tuesday, September 15, 2020 7:49 PM

say $X(t)$ is a r. p.

So at time t_0, $E[X(t_0)]$ is well-defined, if $X(t_0)$ has a mean

just a random variable

So $\boxed{\underline{\text{def.}}\quad \mu_X(t) = E[X(t)]}$

or $\mu_X(t_0) = E[X(t_0)]\ \forall t_0 \in \mathbb{R}$

$$\boxed{\text{def.}\quad \sigma_X^2(t) = E\left[\left(X(t) - \mu_X(t)\right)^2\right]}$$

Ex. $X(t) = A\cos(\omega t + \Theta)$, $\Theta = U(0, 2\pi)$,

$$\mu_X(t) = \int_0^{2\pi} A\cos(\omega t + \theta)\left(\tfrac{1}{2\pi}\right)d\theta = 0 \quad \text{const!}$$

Tuesday, September 15, 2020 7:54 PM

Def The autocorrelation function of r.p. $X(t)$ to be

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)]$$

$$= \iint x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1\,dx_2$$

just $r_{X(t_1)X(t_2)}$

Ex. $X(t) = A\cos(\omega t + \Theta)$ $\Theta = U(0, 2\pi)$

$$R_X(t_1,t_2) = E[(A\cos(\omega t_1 + \Theta))(A\cos(\omega t_2 + \Theta))]$$

$$= \frac{A^2}{2} E[\cos(\omega(t_1 - t_2)) + \cos(\omega t_1 + \omega t_2 + 2\Theta)]$$

$$= \frac{A^2}{2}\cos(\omega(t_1 - t_2))$$

depends only on $t_1 - t_2$ the time difference

Def A process $X(t)$ is Wide Sense Stationary (WSS) if

1) $\mu_X(t)$ is indep. of t and

2) $R_X(t_1, t_2) = R_X(\tau)$ is a function of only $t_1 - t_2 = \tau$

So $A\cos(\omega t + \Theta)$ was WSS!

Tuesday, September 15, 2020 8:06 PM

because for WSS, $R_X(t_1, t_2)$ depends only on delay, we have

$$R_X(t_1, t_2) = R_X(t_2, t_1)$$
$$R_X(\tau) = R_X(-\tau)$$

for WSS R_X is even in τ

Def A process is <u>cyclostationary</u> if μ_X and $R_X(t_1, t_2) = R_X(t+\tau, t)$ are periodic with <u>same</u> period T_0

$$\boxed{\text{i.e.} \quad \mu_X(t+T_0) = \mu_X(t) \qquad R_X(t+\tau+T_0, t+T_0) = R_X(t+\tau, t)}$$

or

No one writes like this $\rightarrow$ $R_X(t_1+T_0, t_2+T_0) = R_X(t_1, t_2)$

Ex. $X(t) = A\cos\omega t + AN(t)$ where $N(t)$ is an indep. sample of $N(0,1)$ at each t

$$\mu_X(t) = E[A\cos\omega t + AN(t)]$$
$$= A\cos\omega t \text{ , periodic w/ period } 2\pi/\omega$$

$$R_X(t+\tau, t) = E\left[\left(A\cos(\omega t + \omega\tau) + AN(t+\tau)\right)\left(A\cos(\omega t) + AN(t)\right)\right]$$

$$= A^2 E\left[\cos(\omega(t+\tau))\cos\omega t + N(t)\cos(\omega(t+\tau)) + N(t+\tau)\cos(\omega t) + N(t)N(t+\tau)\right]$$

$$= A^2\left(\cos(\omega(t+\tau))\cos\omega t + 0 + 0 + E[\cancelto{0}{N(t)}]\,E[\cancelto{0}{N(t+\tau)}]\right)$$

$$= A^2 \cos\omega t \cos(\omega(t+\tau))$$ periodic w/ period $2\pi/\omega$

X cyclostationary

Tuesday, September 15, 2020 8:18 PM

For X cyclostationary, we define

$$\overline{R_X}(\tau) = \frac{1}{T_0}\int_0^{T_0} R_X(t+\tau, t)\,dt$$

"average autocorrelation"

function of only the delay

Ex. $A(\cos\omega t + N(t))$

$$R_X(t+\tau, t) = A^2 \cos\omega t \cos(\omega(t+\tau)), \quad T = 2\pi/\omega$$

$$\overline{R_X}(\tau) = \frac{A^2}{T}\int_0^{T} \cos\omega t \cos(\omega(t+\tau))\,dt$$

$$= \frac{A^2}{2T}\int_0^{T} \cos(\omega\tau) + \cancelto{0}{\cos(2\omega t + \omega\tau)}\,dt$$

$$= \frac{A^2}{2}\cos\omega\tau$$

Tuesday, September 15, 2020 8:23 PM

Def $X(t), Y(t)$ indep. if $\forall m, n \in \mathbb{N}$

$\forall\ t_1, \dots, t_m, \tau_1, \dots, \tau_n \in \mathbb{R}$ we have

$(X(t_1), \dots, X(t_m))$ and $(Y(\tau_1), \dots, Y(\tau_n))$ are indep

$X(t), Y(t)$ indep $\Rightarrow$ $X(t_1), Y(t_2)$ are indep as r.v.s

Def Cross correlation b/t $X(t), Y(t)$ by

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

indep $\Rightarrow$ uncorrelated

Tuesday, September 15, 2020 8:26 PM

Def $X(t), Y(t)$ are jointly WSS if X, Y are each WSS and

$R_{XY}(t_1, t_2)$ dep. only on $t_1 - t_2 = \tau$

So we have $R_{XY}(\tau)$

Tuesday, September 15, 2020 8:27 PM

Let's get back to signals/systems

LTI system w/ impulse resp. h

input r.p. $X(t)$

$$X(t) \to \boxed{h} \to Y(t) = X(t) * h$$

<u>Claim</u> If X is WSS, then Y and X are jointly WSS

and $$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)\,dt$$

$$R_{XY}(\tau) = R_X(\tau) * h(-\tau)$$

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

<u>Proof</u> (see text or ... later)

Tuesday, September 15, 2020 8:32 PM

Ex. $h(t) = \delta(t - t_0)$ delay

$$x \to \boxed{h} \to y$$

$$y(t) = X(t - t_0)$$

now — input $X(t)$ WSS

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = \mu_X$$

$$R_{XY} = R_X(\tau) * \delta(-\tau - t_0) = R_X(\tau + t_0)$$

$$R_Y = R_X(\tau) * \delta(-\tau - t_0) * \delta(\tau - t_0)$$

$$= R_X(\tau + t_0) * \delta(\tau - t_0)$$

$$= R_X(\tau + t_0 - t_0) = R_X(\tau)$$

Tuesday, September 15, 2020 8:36 PM

Ex. $h(t) = \frac{1}{\pi t}$ ← H. T.

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} \frac{1}{\pi t} dt = 0 \quad \text{(odd function)}$$

$$R_{XY}(\tau) = R_X(\tau) * \frac{1}{-\pi t} = -\hat{R}_X(\tau)$$

$$R_Y(\tau) = R_X(\tau) * \frac{1}{-\pi t} * \frac{1}{\pi t} = -\hat{\hat{R}}_X(\tau)$$

$$= R_X(\tau)$$

Tuesday, September 15, 2020 8:38 PM

Freq. Domain?

Define an analogue to F.T.

Power Spectral Density gives the power of a random process at diff. frequencies

Def. $S_X(\omega) = E\left[|X(\omega)|^2 \right]$

THM (Wiener-Khinchin)

For a WSS process, the P.S.D. is given by

$$S_X(\omega) = \mathcal{F}\{R_X(\tau)\}(\omega)$$

For a cyclostationary process,

$$S_X(\omega) = \mathcal{F}\{\overline{R}_X(\tau)\}(\omega)$$

This is the F.T. for random signals

Tuesday, September 15, 2020 8:43 PM

The power content of a process is def. by

$$P_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\, d\omega$$

---

if a process is WSS, then

$$R_x = \mathcal{F}^{-1}\{S_x(\omega)\}$$

so $R_x(\tau) = \int S_x(\omega) e^{j\omega t}\, d\omega$

so $R_x(0) = \int S_x(\omega)\, d\omega = P_x$

$$\boxed{\text{for WSS, } P_x = R_x(0)}$$

Tuesday, September 15, 2020 8:46 PM

$$X(t) \to \boxed{h} \to Y(t)$$

WSS LTI

$\mu_Y = \mu_X \int h$,

$R_Y = R_X * h(-\tau) * h(\tau)$

$\downarrow$ FT both sides

$$\boxed{S_Y(\omega) = |H(\omega)|^2 S_X(\omega)}$$ ← use this a lot

$\mathcal{F}\{R_X\} = \mathcal{F}\{S_{XX}\}$

$$\boxed{\mu_Y = \mu_X H(0)}$$