Tuesday, October 6, 2020 5:19 PM

A → D • quantization quantified by SQNR

- optimal quantization
- Important for comm schemes with digital backends analog info

What if TXing digital information?

↳ What does this really mean?

transmission always occurrs in cts-time

I am trying to both transmit and receive data which is already quantized (a string of bits)

→ quantification of noise/loss is very different (rate of bit errors)

→ do not care about SQNR

Basics

Tuesday, October 6, 2020 6:11 PM

Multiple binary coding schemes

"Natural Binary Code" (NBC) N levels, 0 → N-1

0000 → 0
0001 → 1
0010 → 2
⋮
1111 → 15

Gray Coding: Two successive values differ by only one bit.

3-bit:

0 → 000
1 → 001
2 → 011
3 → 010
4 → 110
5 → 111
6 → 101
7 → 100

In NBC b-bits, 2^{b-1} is the largest error I can get from 1 bit error

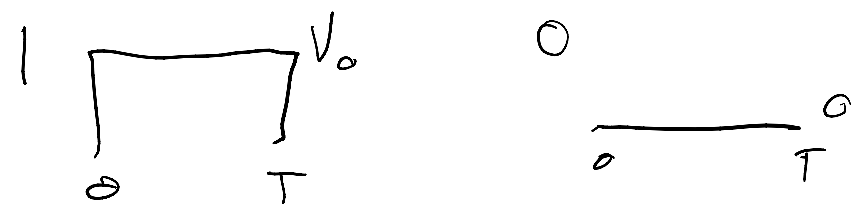
In Gray, 1 is the largest (indep. of b)

Tuesday, October 6, 2020 6:16 PM

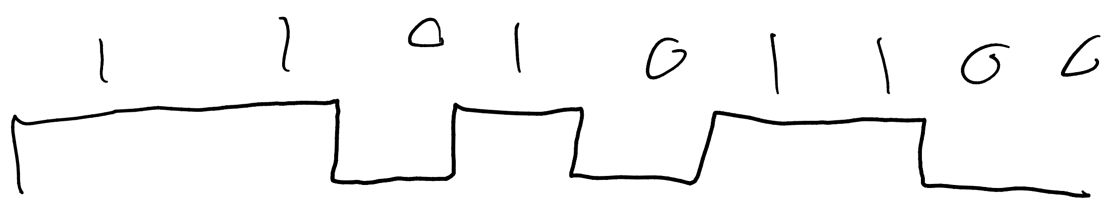
Problem: I want to transmit my coded 1s and 0s but the world is **Analog**!

DLD solution Channel: a wire

Scheme:



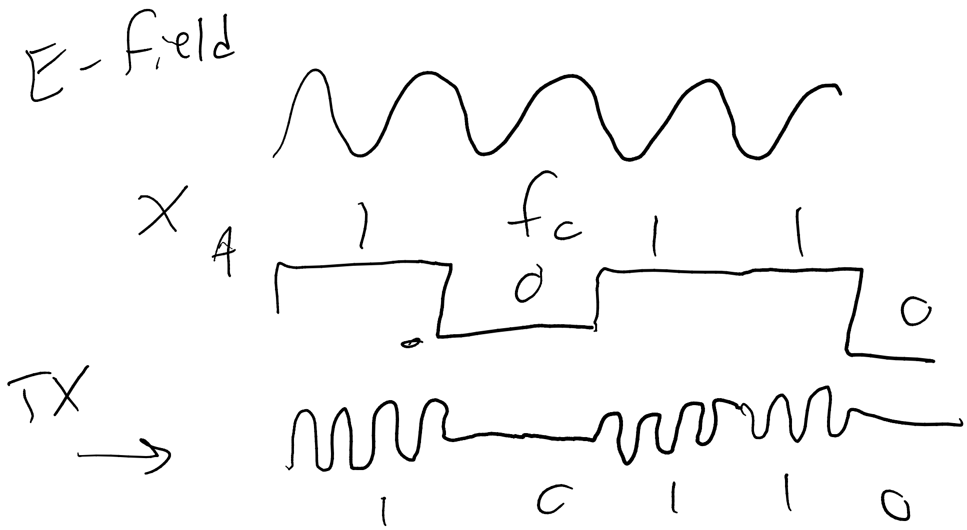
signal



cts-time signal representing a **digital** signal.

How about via E+M waves?

simple scheme:


E-field
x
fc
A
1 0 1 1 0
TX
1 0 1 1 0

Tuesday, October 6, 2020 6:24 PM

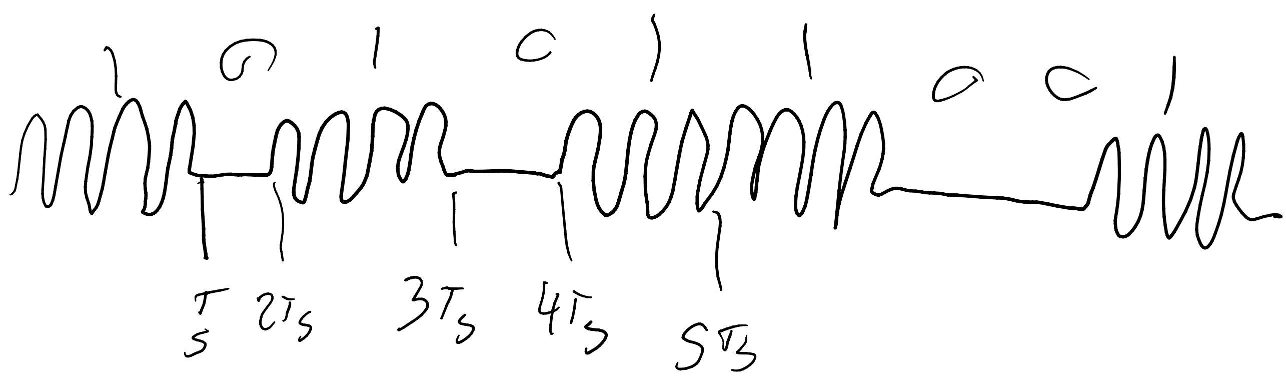
Digital data trying to send over IR

$\lambda \sim 1100 nm$

$c = 3 \times 10^8 m/s$ $f \approx 270 THz$

On-off keying (OOK)

envelope which is 0 over intervals of length T_s represent 0s, A over intervals of length T_s rep. 1s



On-off keying - very natural, useful when TX mechanism has very high freq.

you only need the incoming Power as a fncn of time

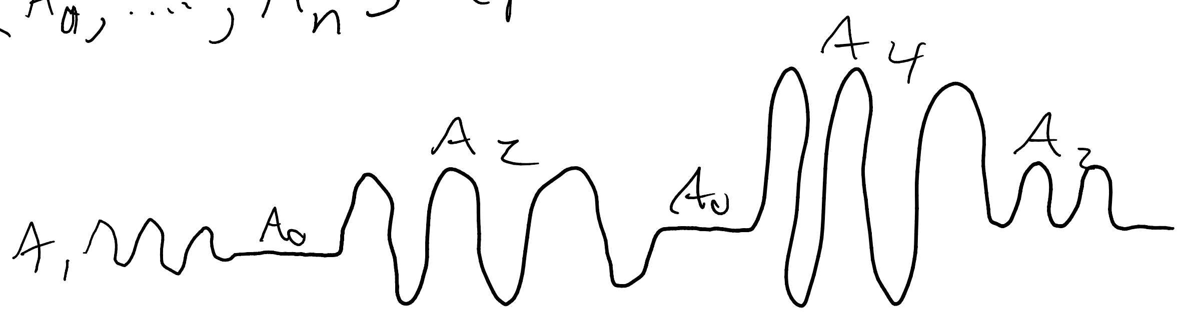
~~X~~ phase info not needed. rate 1 bit/T_s

Amplitude Shift keying

Generalization of OOK with multiple levels

Generalization of OOK with multiple levels

$\{A_0, \dots, A_n\}$ represent ~~n+1~~ different bit-sequences



each A_i level (and thus each pulse of T_S duration) can represent $\log_2(n+1)$ bits

rate $\log_2(n+1)$ bits$/T_S$

If I want to represent X values, how many bits do I need?

A: $\log_2 X$

$\underbrace{}_{b_0}\ \underbrace{}_{b_1}\ \dots\ \underbrace{}_{b_{N-1}}$

2^N values with N bits eg. 8 bits $\to 0 \to 255$

N values rep. by $\log_2 N$ bits

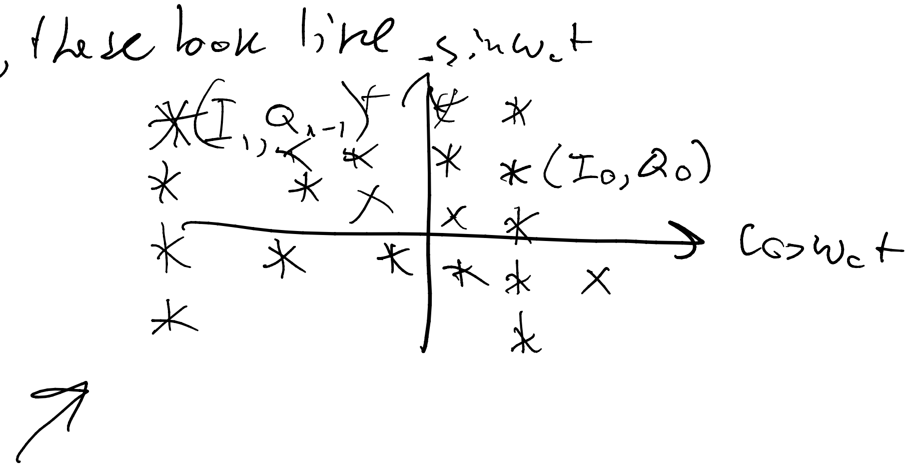
$\log_2 X$ bits rep. $2^{\log_2 X}$ values $= X$ values

RF transmission, where I send

$$I_i \cos w_c t - Q_j \sin w_c t$$

$$\{I_0, \ldots, I_{n-1}\}, \quad \{Q_0, \ldots, Q_{m-1}\}$$

as phasors, these look like



Constellation,

signal $I_i \cos w_c t - Q_j \sin w_c t$

distinct from others only if both amp. and phase taken into account

$n \cdot m$ possible (I, Q) pairs

each signal in const. represents $\log_2(nm)$ bit string

Constellations $\rightarrow$ many comm schemes

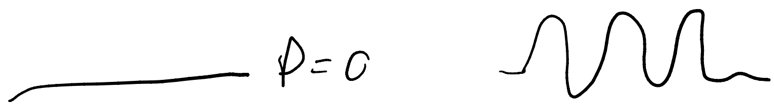
Tuesday, October 6, 2020 6:45 PM

Before (AM): Learn a new mod scheme

→ first question – how does noise impact this?

... How do we <u>decide</u> if we receive a certain signal?

<u>OOK case</u>



$P=0$ $P=P_0$

I register incoming power of a pulse $P_r \neq 0$

$\neq P_0$

because of noise

which signal was it?

in this case w/ 2 possibilities I might say

"received 1" if $P_r > T$, some threshold

"received 0" else

What about for ASK?

less simple – I might need N thresholds

"received A_0" if $0 \leq P_r < T_0$

"received A_1" if $T_0 \leq P_r < T_1$

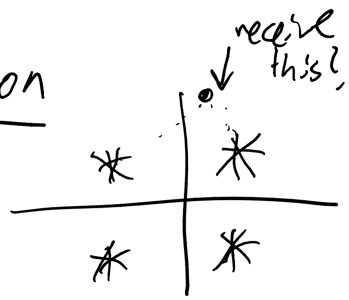
⋮

etc

⋮

etc

For constellation



a natural guess is L^2 distance min

generally: need 2D decision regions

"received A_i" if $r(t) \in \mathcal{R}_i \subset \mathbb{R}^2$

in all 3 cases: need to pick decision regions

Q: What are the best decision regions?

Tuesday, October 6, 2020 6:54 PM

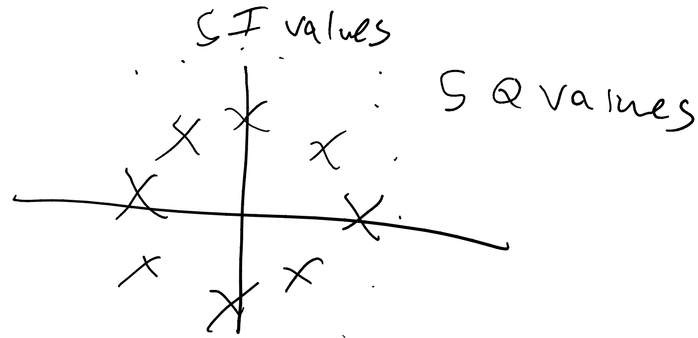
$I_i \cos\omega_c t - Q_j \sin\omega_c t$

$\{I_0, \dots, I_{n-1}\}, \{Q_0, \dots, Q_{m-1}\}$

if $\{I, Q\} = \{I_0, \dots, I_{n-1}\} \times \{Q_0, \dots, Q_{m-1}\}$

I get all ordered pairs → rectangular const.

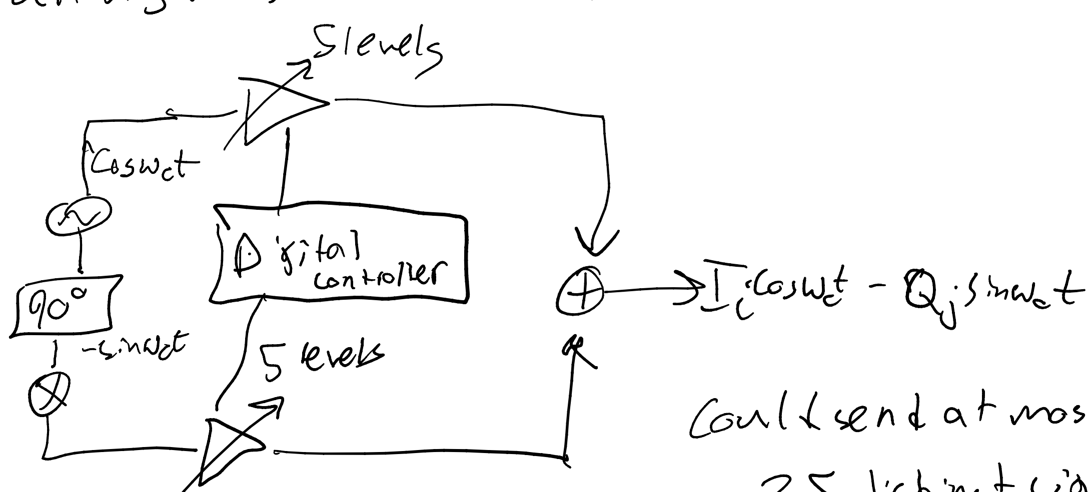
Could restrict $\{I, Q\}$ to a subset of this product



but only 8 total const. points

represent fewer bitstrings than a rectangle

but all signals have same $|\cdot|$



Could send at most

25 distinct signals

but can choose to send

≥ 2 distinct signals

but can choose to send fewer

Decision Theory

Tuesday, October 6, 2020 6:53 PM

Mathematical context

our signals are represented by **basis vectors**

In OOK, $s_1(t) = A\cos\omega_c t, \quad 0 \le t \le T_c$

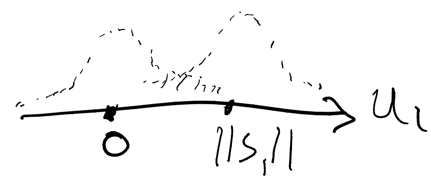
$s_0(t) = 0, \quad 0 \le t \le T_c$

lives in a 1-dimensional space

basis vector: $\frac{s_1(t)}{\|s_1(t)\|} = u_1$ ← unit energy

L^2 norm

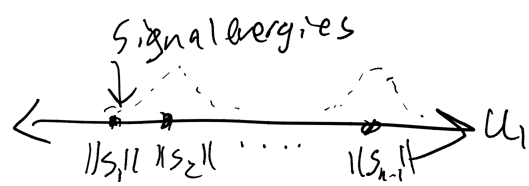
$s_0 = 0 \cdot u_1, \quad s_1 = \|s_1\| u_1$



In ASK, $s_i(t) = A_i \cos\omega_c t, \quad 0 \le t \le T_c$

$u_1 = \frac{s_i}{\|s_i\|}$ for any s_i

$s_i = \|s_i\| u_i$



OOK special case
lives geometrically

"nearness" geometrically = susceptibility to noise

In constellations – same idea but 2-dim

← I, Q already define dims

In constellations – same ...

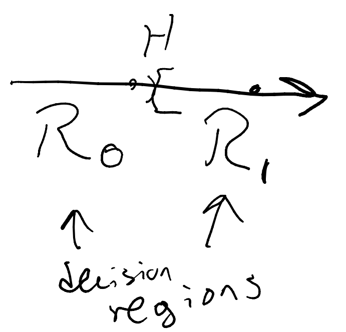
(I, Q already define dims already 2 orthogonal signal components)

"nearness" is precisely $\mathcal{L}^2$ distance

Constrain to OOK

1^{st} Strategy to decide decision regions minimize Prob. of a mistake

$$P[\text{mistake}] = P[\text{guess } 1 \mid TX\ 0] + P[\text{guess } 0 \mid TX\ 1]$$



receive $Y = \int_0^T |y(t)|^2 dt \in \mathbb{R}$

Y is either $\geq$ or $<$ H my threshold

$$P[\text{error}] = P[Y \in R_1 \mid \underset{\hat{x}}{I} \text{ transmitted } 0] + P[Y \in R_2 \mid I \text{ transmitted a } 1]$$

$$= \int_{R_1} f_{Y,\hat{x}=0}(y)\,dy + \int_{R_0} f_{Y,\hat{x}=1}(y)\,dy$$

our decision algorithm should say

if $P[y \text{ and } \hat{x}=0] > P[y \text{ and } \hat{x}=1]$

then decide 0

happens if $y \in R_0$

more generally for K classes

$$P[y \text{ and } \hat{x}=x_i] > P[y \text{ and } \hat{x}=x_j] \quad \forall j \neq i$$

max. $P[y \text{ and } \hat{x}=x_i]$ over i

$$P[y \text{ and } \hat{x}=x_i] = P[\hat{x}=x_i \mid y]\, P[y]$$

$P[y]$ is common to this term for each i,

max $P[y \text{ and } \hat{x}=x_i] \Longleftrightarrow \boxed{\max P[\hat{x}=x_i \mid y]}$

posterior prob.

maximum a posteriori (MAP) method -

maximize $P[\hat{x}=x_i \mid y]$

$R_i^{MAP} = \{ y \mid P[\hat{x}=x_i \mid y] > P[\hat{x}=x_j \mid y] \ \forall j \neq i \}$

$$\boxed{R_i^{MAP} = \{ y \mid P[\hat{x} = x_i \mid y] \geq P[\hat{x} = x_j \mid y] \ \forall j \neq i \}}$$

($\hat{x}$ is the ground truth, the real transmitted signal
y is received - x corrupted by noise)

$\boxed{\text{MAP is the method which minimizes the } P_{error}}$

If I know I receive a 1 only 0.005% of the time, then

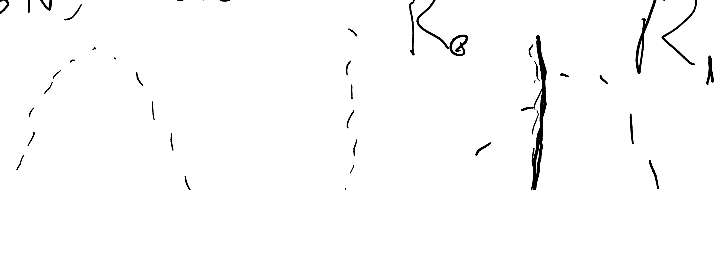
$$P[\hat{x} = 0 \mid y] = \frac{P[y \mid \hat{x} = 0] P[\hat{x} = 0]}{P[\hat{x} = 1] P[y \mid \hat{x} = 1] + P[\hat{x} = 0] P[y \mid \hat{x} = 0]}$$

$$P[\hat{x} = 1 \mid y] = \frac{P[y \mid \hat{x} = 1] P[\hat{x} = 1]}{\text{same denom}}$$

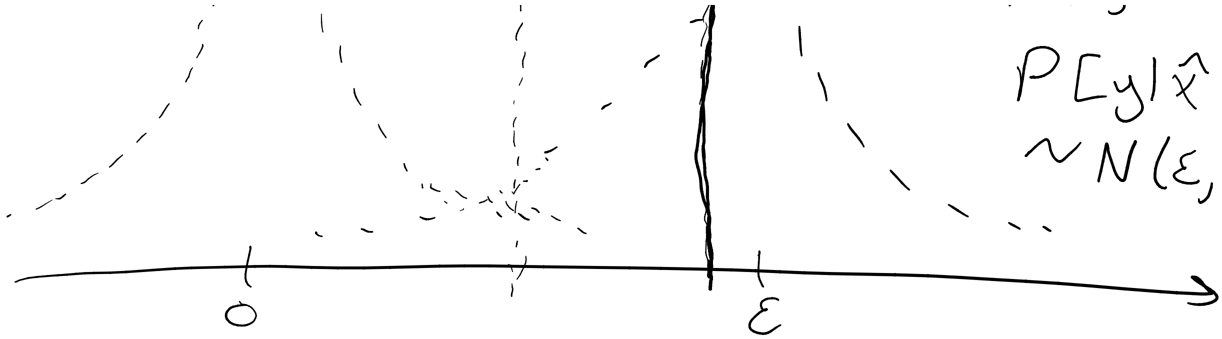
maxing LHS is eq. to maxing $P[y \mid \hat{x} = x_i] P[\hat{x} = x_i]$

in example, $P[\hat{x} = 1]$ is <u>very low</u> so this will greatly affect decision region

AWGN, σ^2_{noise}



$P[y \mid \hat{x} = 0]$
$\sim N(0, \sigma^2)$
$P[y \mid \hat{x} = 1]$



$P[y|\hat{x}=1] \sim N(\varepsilon, \sigma^2)$

$P[\hat{x}=0] >>> P[\hat{x}=1]$

So MAP minimizes P_{error} but requires we **know**, a priori, $P[\hat{x}=x_i]\ \forall x_i$

Why - Bayes' THM

What if we don't know? Can't do MAP! (or, rather, we have to assume priors $P[\hat{x}=x_i]$)

Instead - we look to maximize

$$P[y \mid \hat{x}=x_i]$$ ← different conditional order than MAP

likelihood fn

Prior

Bayes' THM $$P[\hat{x}=x_i \mid y] = \frac{P[y|\hat{x}=x_i]\, P[\hat{x}=x_i]}{\sim}$$

posterior

Call this method **ML** or maximum likelihood call this method ML or Maximum like...

$$\boxed{\mathcal{R}_i^{ML} = \{ y \mid P[y \mid \hat{x} = x_i] \geq P[\hat{x} = x_j] \ \forall j \neq i \}}$$

note: $\max P[y \mid \hat{x} = x_i]$ (ML)

is same as $\max P[\hat{x} = x_i \mid y] = \underbrace{P[y \mid \hat{x} = x_i] P[\hat{x} = x_i]}$ (MAP)

if $P[\hat{x} = x_i]$ is i-independent

(i.e, equiprobable events)

$$\boxed{\text{MAP} = \text{ML} \iff \text{all } x_i \text{ are equiprobable}}$$

in BOK example — to determine MAP,

need $P[\hat{x} = 0]$, $P[\hat{x} = 1]$ and numerics regarding the tail distributions of the Gaussians to determine $\mathcal{R}_0, \mathcal{R}_1$

to determine ML — H = midpoint.

$\varepsilon/2$

ML — "intuitive guess" knowing nothing

MAP — "sophisticated derivation" knowing something

"Minimize expected loss" $L(x_i, x_j)$ = loss of mistaking x_i for x_j

$$\mathbb{E}[L] = \sum_i \sum_j \int_{R_j} L(x_i, x_j)\, P[y \text{ and } \hat{x} = x_k]\, dy$$

another decision method (not used here)

use SGD to minimize this function

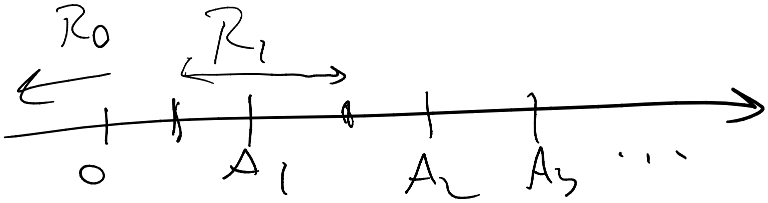
A special case that does matter:

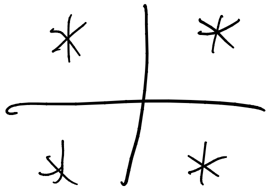
<u>minimum square error</u> (geometric idea)

Square error: $\|y - \hat{x}_i\|^2$

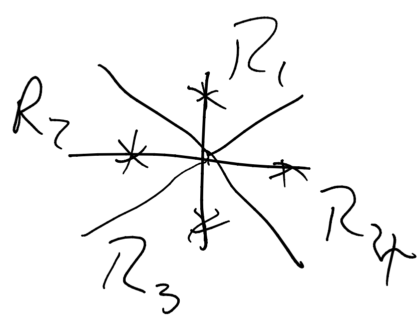
$$\boxed{R_i^{LSE} = \{ y \mid \|y - \hat{x}_i\|^2 \le \|y - x_j\|^2 \ \forall j \ne i \}}$$

no <u>probability</u> needed   purely geometrical





quadrants are R_i



super easy method

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in AWGN case,

ML: $t = E/2$

in AWGN case,

ML: $T = \mathcal{E}/2$
LSE: $H = \mathcal{E}/2$

for AWGN, if signals are "equidistant"

*obvious

$$ML = LS$$

stronger: $\boxed{\text{AWGN} \Rightarrow ML = LS}$

$$\boxed{\text{AWGN + equiprobable} \Rightarrow ML = LS = MAP}$$

Last rule: <u>Correlation Receiver</u> or <u>Matched Filter</u>

Maximize innerproduct $\langle y, \hat{x}_i \rangle$ ← $\mathcal{L}_2$ i.p.

$$\int y(t)\, \hat{x}_i(t)\, dt = \langle y, \hat{x}_i \rangle \quad \leftarrow \text{real case}$$

or more gen.

$$\max \mathrm{Re}\langle y, \hat{x}_i \rangle = \mathrm{Re}\left(\int y(t)\, \hat{x}_i^*(t)\, dt \right)$$

$$\mathcal{R}_i^{MF} = \{ u \mid \mathrm{Re}\langle u, x_i \rangle > \mathrm{Re}\langle u, x_j \rangle \ \forall j \neq i \}$$

$$\mathcal{R}_i^{ML} = \{ y \mid Re\langle y, x_i\rangle \geq Re\langle y, x_j\rangle \ \forall_j \neq i \}$$

$\hat{x}_0 =$

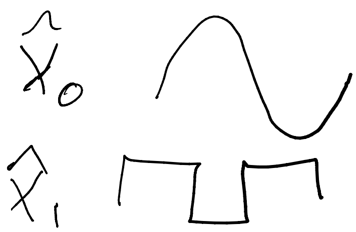
$\int 0 = 0$

$\hat{x}_1 =$

$\int A = \varepsilon \geq 0$ wow, always pick A

weird method

what if my scheme involved sending different pulse shapes?

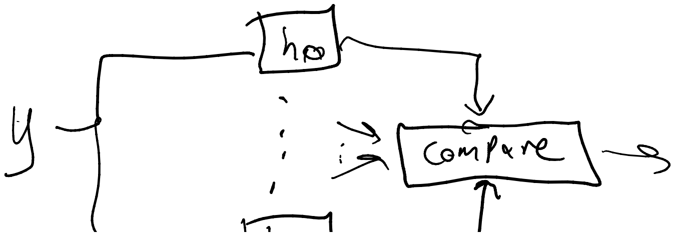


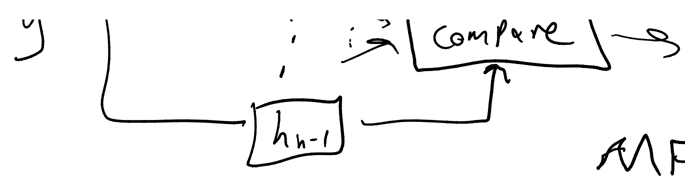
max corr. occurs if you are basically integrating the square of the signal

<u>So long as</u> $\hat{x}_0$ and $\hat{x}_1$ have equal energy

Corr. receiver <u>makes sense</u> if equal energy symbols

$h_i = \hat{x}(T_s - t)$ then $\langle y, \hat{x}\rangle = y * h_i$





MF is very easy in hardware

equal energy

$$E = \int \hat{x}_i(t)^2 dt \quad \forall i$$

then

$$SE = \int \left(y(t) - \hat{x}_i(t)\right)^2 dt = \underbrace{\int y^2 dt + E^2}_{i\text{-indep.}} - 2 \underbrace{\int y \hat{x} dt}_{\langle y, \hat{f} \rangle}$$

So max. MF = min. LS

$$\boxed{\text{Equal energy} \Rightarrow MF = LS}$$

$$\boxed{\begin{array}{c}\text{Equal energy} \\ + \\ \text{equiprobable} \\ + \\ \text{AWGN}\end{array} \Longrightarrow MF = LS = ML = MAP}$$

MF = easiest hardware imp

under these assumptions

get min. P. of error! (great :))

Binary Signaling

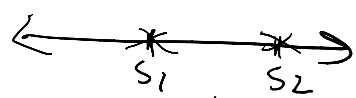
Tuesday, October 6, 2020 8:12 PM

Def A binary signaling scheme is one in which only 2 symbols are used, $s_1(t)$ and $s_2(t)$

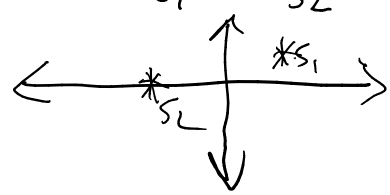
→ each symbol can represent only 1 bit of information

binary signalling schemes can live in either 1-D or 2-D

1D case $K s_1(t) = s_2(t)$



2D case $k s_1(t) \neq s_2(t)$ for any k

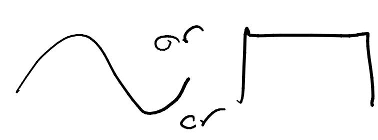


$$\frac{s_1(t)}{\|s_1(t)\|} = \psi_1$$ has unit energy, it's our first basis vector

$$\int |s_1(t)|^2 dt = \mathcal{E} \quad \text{but} \quad \|s_1(t)\| = \sqrt{\int s_1(t)^2 dt} = \sqrt{\mathcal{E}}$$

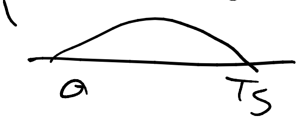
$$\psi_1 = \frac{s_1(t)}{\sqrt{\mathcal{E}_{s_1}}}$$

Binary antipodal

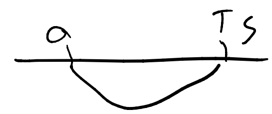


Binary antipodal

$1: s_1(t) = p(t), 0 \le t \le T_s$ "sine pulse"



$0: s_0(t) = -p(t), 0 \le t \le T_s$

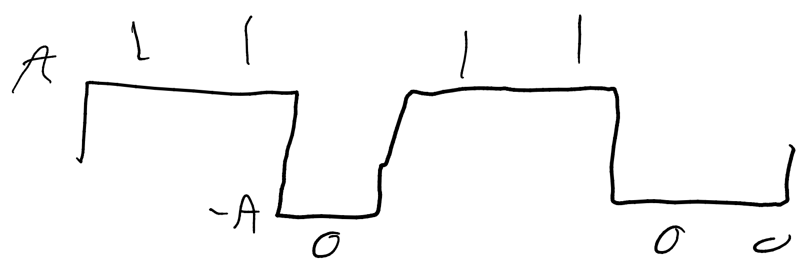


immediately clear that $\mathcal{E}_{s_1} = \mathcal{E}_{s_0} = \int |p(t)|^2 dt$

energy per bit $\mathcal{E}_b = \mathcal{E}_s / \#\text{bits rep. by } s \underset{\text{binary signaling}}{=} \mathcal{E}_s$

$s_1(t) = A, 0 \le t \le T_s$

$s_0(t) = -A, 0 \le t \le T_s$

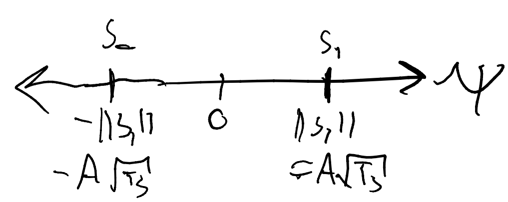


"shifted DLD" but equal energy

$$\mathcal{E}_b = \int_0^{T_s} A^2 \, dt = A^2 T_s$$

"Binary Pulse Amplitude Modulation"
or binary PAM

$$\psi_1 = \frac{s_1}{\|s_1\|} = \frac{A}{\sqrt{A^2 T_s}} = \frac{1}{\sqrt{T_s}}$$



Binary amplitude Shift keying

$S_1(t) = p(t)\cos \omega_c t \quad 0 \le t \le T_s$

$S_0(t) = -p(t)\cos \omega_c t \quad 0 \le t \le T_s$

must be infinite bandwidth

b/c defined for finite time

Facilitate FDM

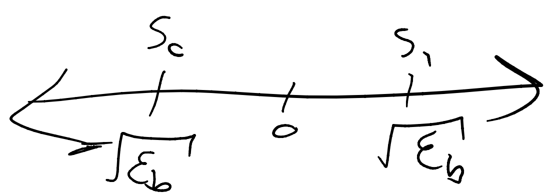
can send on RF

if $p(t) = A$ (rectangular)

$$S_1(t) = \sqrt{\frac{2\mathcal{E}_b}{T_s}}\cos \omega_c t$$

$$S_2(t) = -S_1(t)$$

$$\psi_1(t) = \sqrt{\frac{2}{T_s}}\cos \omega_c t, \quad 0 \le t < T_s$$



just another antipodal method

antipodal is the only equal energy 1-D signaling scheme

Binary signaling in 2D can be equal energy in many ways but

most common is binary orthogonal signalling

$1 \to s_1(t)$

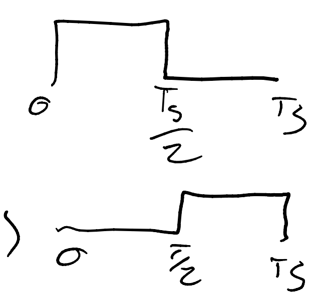
$0 \to s_0(t)$

$$\int_0^{T_s} s_1(t)\, s_0(t)\, dt = 0$$

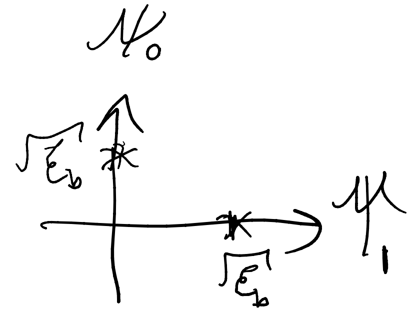
ex:

$s_1(t)$

$s_0(t)$

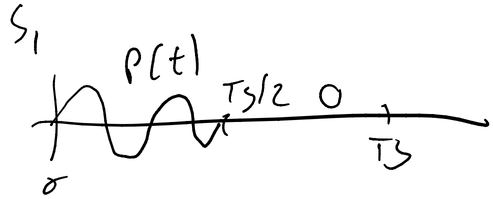


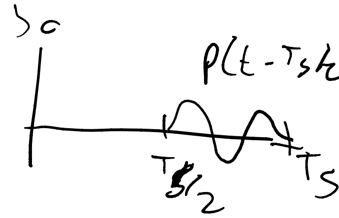
$$\psi_1 = \frac{s_1}{\|s_1\|}, \quad \psi_0 = \frac{s_0}{\|s_0\|}$$



$s_1 = (\sqrt{E_b}, 0)$, $s_0 = (0, \sqrt{E_b})$ in this basis

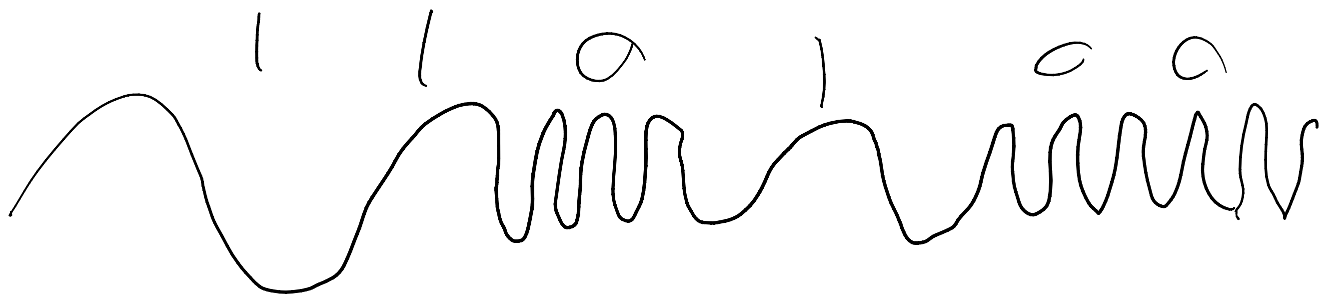
Binary pulse position modulation





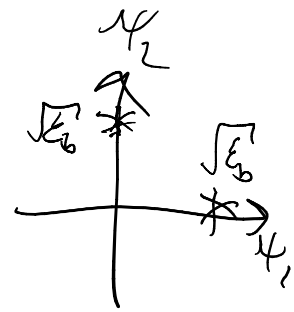
Binary Frequency Shift Keying (FSK)

$$s_1(t) = \sqrt{\frac{2E_b}{T_s}} \cos \omega_1 t, \quad 0 \le t \le T_s \qquad s_2(t) = \sqrt{\frac{2E_b}{T_s}} \cos \omega_2 t, \quad 0 \le t \le T_s$$

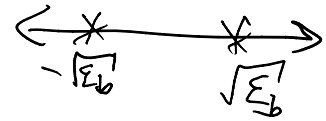


"like FM but digital"

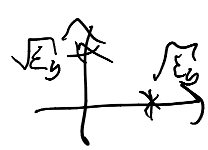
$$\psi_1 = \sqrt{\frac{2}{T_s}} \cos \omega_1 t, \quad \psi_2 = \sqrt{\frac{2}{T_s}} \cos \omega_2 t$$



Binary signaling is either 2 antipodal signals



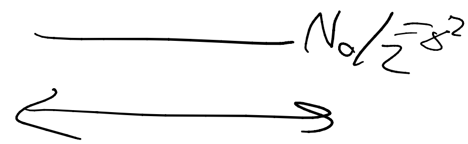
or 2 orth. signals



Tuesday, October 6, 2020 8:41 PM

AWGN

model: $s_i(t) + n(t)$, n has PSD



binary antipodal

$$s_0(t) = -\sqrt{\mathcal{E}_b}\,\psi(t)$$

$$s_1(t) = \sqrt{\mathcal{E}_b}\,\psi(t)$$

for some ψ s.t.
$\psi(t) = 0$ for $t \notin (0, T_s)$
and $\|\psi(t)\| = 1$

receive $r(t) = \pm\sqrt{\mathcal{E}_b}\,\psi(t) + n(t)$

need to make decision as to whether r is probably s_0 or s_1

- AWGN
- Equal energy

$\longrightarrow$ MF = LS = ML

(if equiprobable = MAP as well)

we can use MF

"integrate over T_s"

$r(t) \longrightarrow \boxed{\psi(T_s - t)} \xrightarrow{t = T_s} \longrightarrow u(T_s)$

$$r(t) \longrightarrow \boxed{\psi(T_s - t)} \longrightarrow \; \diagup \; \longrightarrow y(T_s)$$

measuring at T_s

$$y(T_s) = \int_0^{T_s} r(\tau)\,\psi(\tau - T_s \cancel{+ T_s})\,d\tau$$

$$= \int_0^{T_s} r(\tau)\,\psi(\tau)\,d\tau$$

$$= \pm\sqrt{\mathcal{E}_b}\,\underbrace{\int_0^{T_s} \psi^2(\tau)\,d\tau}_{1} + \int_0^{T_s} n(t)\,\psi(t)\,dt$$

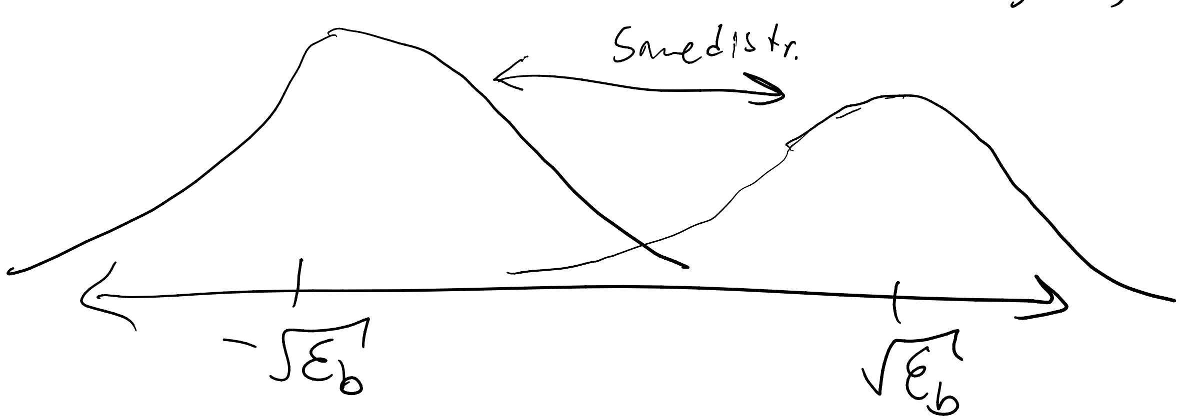
$$= \pm\sqrt{\mathcal{E}_b} + \underbrace{\int_0^{T_s} n(t)\,\psi(t)\,dt}_{\text{filtered white noise}}$$

same mean -0

can show $-$ same σ^2 because $\|\psi\| = 1$

$$= s_i(t) + n$$

distributed like $N(0, \sigma^2)$



ideal decision region in MF = LS = ML is just

Ideal decision region in MF = LS : ML is just threshold at $\underline{0}$

indep. of ψ

clear that for same N_0,

$\nearrow \mathcal{E}_b \quad \searrow P_{err}$ as the tail "disappears" as you get further from 0