Tuesday, October 13, 2020 3:02 PM

$\{s_i(t)\}_{i=1}^{2^n}$ which represent 2^n different pieces of information (encoded as n-bit strings)

By Gram-Schmidt, we can represent these signals as a lin. comb. of $\{\varphi_i\}_{i=1}^{m}$ (m dep. on signalling scheme $m \leq 2^n$)

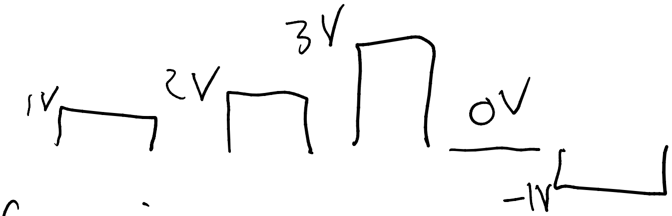
where $\langle \varphi_i, \varphi_j \rangle = \int \varphi_i \varphi_j \, dt = \delta_{ij}$ ← Kronecker delta $= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

<u>Special case 1</u> Orthogonal signalling $\langle s_i, s_j \rangle = 0$ if $i \neq j$

$\{\varphi_i\}_{i=1}^{2^n}$, $\varphi_i = \frac{s_i}{\|s_i\|}$ <u>ex.</u> $\{\cos \omega_i t\}_{i=1}^{2^n}$ $\omega_i \neq \omega_j$ for $i \neq j$

<u>Special case 2</u> 1-dimensional signalling $s_j = \alpha_{ij} s_i$

α is t-independent <u>ex</u>



etc.

$\{\varphi_i\} = \{\varphi\} = \left\{\frac{s_i}{\|s_i\|}\right\}$ for any i

---

In any scheme I can draw <u>constellation</u> in $m < 2^n$ dimensions, where each axis rep. φ_i

dimensions, where each axis rep. φ_i

$$\{s_i\} \subseteq \mathcal{F}(\mathbb{R},\mathbb{R}) \qquad \text{but} \quad s_i = \sum_{j=1}^{m} a_{ij}\varphi_j$$

$$\{\varphi_i\} \subseteq \mathcal{F}(\mathbb{R},\mathbb{R}) \qquad \text{where } \vec{a}_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{im} \end{pmatrix} \in \mathbb{R}^m$$

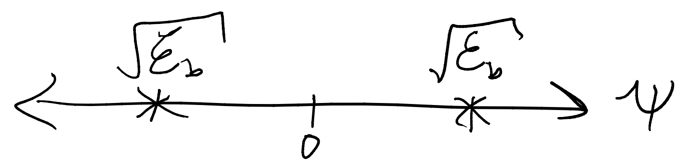
so each signal has a natural representation as an m-vector.

<u>special cases of m=1 or 2</u>, can literally <u>draw</u> on paper

<u>ex.</u> antipodal signalling $s_0(t) = -s_1(t)$

$$\psi = \frac{s_1}{\|s_1\|} \rightarrow s_1 = \|s_1\|\psi = \sqrt{\mathcal{E}_b}\,\psi$$

$$s_2 = -s_1 = -\sqrt{\mathcal{E}_b}\,\psi$$

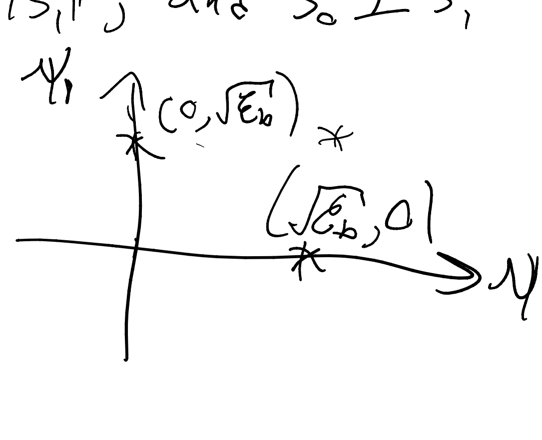


(Binary ASK
$s_1(t) = p(t)\cos\omega_c t$ for some $p(t) = 0$ $\forall t<0, t>T$)

<u>ex.</u> binary orthogonal $\|s_0\| = \|s_1\|$, and $s_0 \perp s_1$

$$\psi_0 = \frac{s_0}{\|s_0\|}, \quad \psi_1 = \frac{s_1}{\|s_1\|}$$

$$s_0 = \sqrt{\mathcal{E}_b}\,\psi_0 + 0\cdot\psi_1$$



$$\mathcal{E} = \int s(t)^2\,dt$$

$$\mathcal{E}_s = \int s(t)^2 dt$$

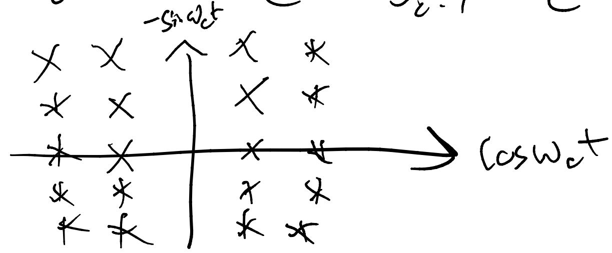
$$||s|| = \sqrt{\int s(t)^2 dt} = \sqrt{\mathcal{E}_s}$$

$$s_2 = \sqrt{\mathcal{E}_b}\,\psi_0 + \sqrt{\mathcal{E}_b}\,\psi_1 = s_0 + s_1$$

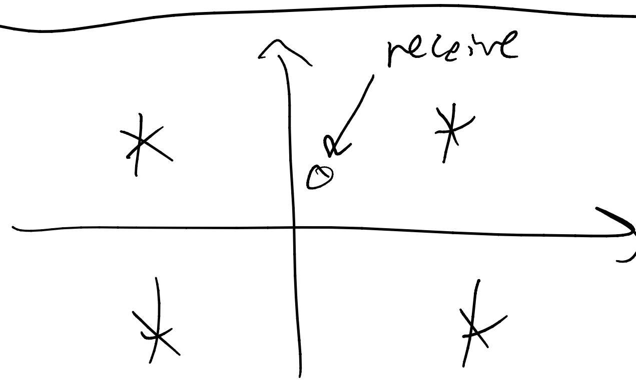
ex.

$I_i \cos w_c t - Q_j \sin w_c t$ $\{I_i\}_{i=1}^N$ $\{Q_j\}_{j=1}^N$

2D constellation w/ multiple signals



Decision theory



never receive exactly one of my signals s_i because of noise. What was r supposed to be?

natural thought w/ constellations in mind: nearest point ($\mathcal{L}^2$)

$\hat{s} = \arg\min_{s_i} ||r - s_i||^2$ signal which minimizes square distance

LS = least squares

LS = least squ...

- didn't consider $P[s=s_i]$, noise, anything really...

Want to think about prob.

minimize P_{error}

maximizing $P[r \text{ and } s=s_i] = P[s=s_i|r]\,P[r]$

I received r and I transmitted s_i

maximize $\boxed{P[s=s_i|r]}$ MAP

maximum a posteriori

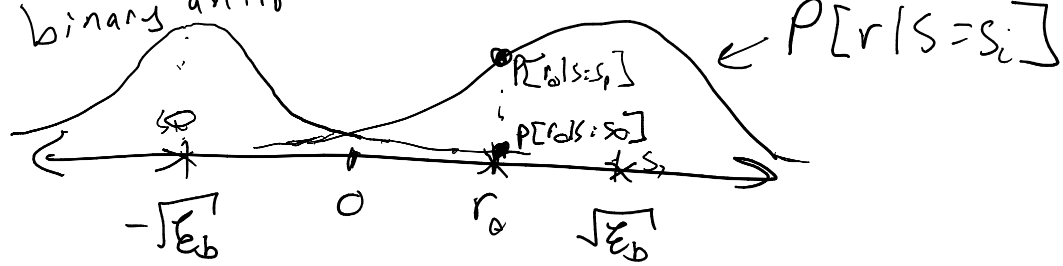
decision regions: $R_i^{MAP} = \{r \mid P[s=s_i|r] \geq P[s=s_j|r]\ \forall j\}$

by Bayes' rule, $P[s=s_i|r] \propto P[r|s=s_i]\,P[s=s_i]$

MAP takes into account $P[s=s_i]$ "priors"

binary antipodal AWGN

ex.



here $P[r_0|s=s_0] << P[r_0|s=s_1]$

here $P[r_0|s=s_0] << P[r_0|s=s_1]$

If I tell you, however, the $P[s=s_0]=0.99$

$P[s=s_1]=0.01$

then maybe $P[r_0|s=s_0]\,P[s=s_0] > P[r_0|s=s_1]\,P[s=s_1]$

this is the effect of prior –

MAP weights the likelihood by the prior to determine decision

issue – may not know priors

assume $P[s_i]=P[s_j]\ \forall i,j$ then MAP is equivalent to maximizing $\boxed{P[r|s=s_i]}$ ML
maximum likelihood

$\boxed{\text{ML} = \text{MAP when all } s_i \text{ equiprobable}}$

$$\mathcal{R}_i^{ML} = \{r \mid P[r|s=s_i] \geq P[r|s=s_j]\ \forall j\}$$

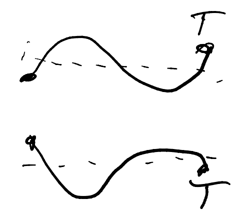
$$\underset{\text{case}}{\text{AWGN}} = \{r \mid \|r-s_i\|^2 > \|r-s_j\|^2\ \forall j\} = \mathcal{R}_i^{LS}$$

$\boxed{\text{LS} = \text{ML in AWGN case}}$

Corr. receiver $\{s_i\}$ maximize "Re" $\langle r, s_i \rangle$

$$= \int_0^T r s_i^* \, dt$$

send many different pulses $\{p_i(t)\}_{i=1}^{N}$



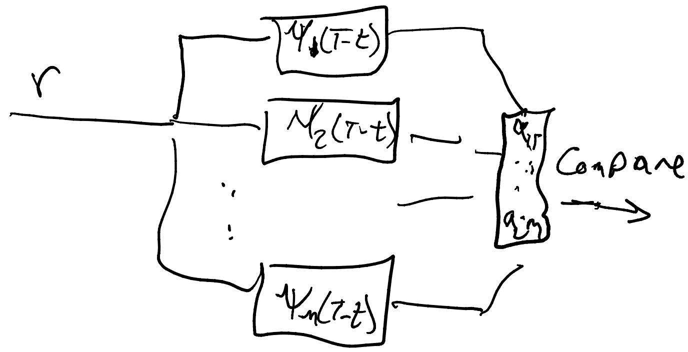
specifically

if $\{s_i\}$ orthogonal, then $\langle s_i, s_j \rangle = 0$ if $i \neq j$

so the only term you'd see in CR is due to noise

$\langle s_i, s_i \rangle = \|s_i\|^2 > 0$ hopefully larger than noise

$\{\psi_i\}_{i=1}^{m}$ rep. $s_i = \sum_{j=1}^{m} a_{ij} \psi_j$



if all pulses have same energy

$$CR = \int_0^T r x_i \, dt$$

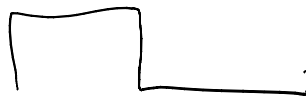
$$LS = \int_0^T (r - x_i)^2 dt = \underbrace{\int r^2 + \int x_i^2}_{\text{indep of } i} + 2\int_0^T r x_i \, dt$$

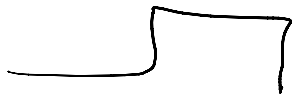
min LS = max CR

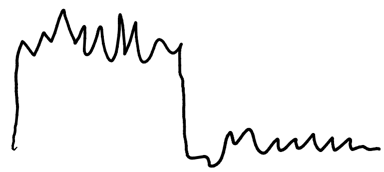
i.e. $\boxed{\text{equal energy} \Rightarrow LS = CR}$

$CR \underset{\substack{\uparrow \\ \text{def}}}{=} MF$ (matched filter)

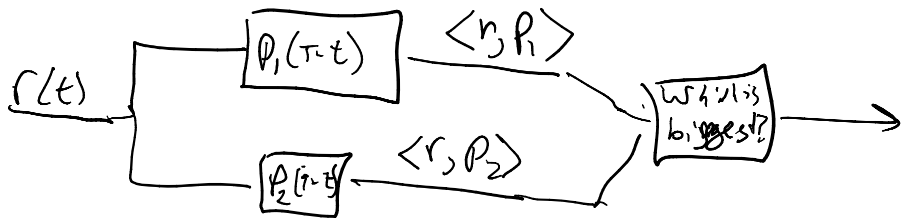
Tuesday, October 13, 2020 3:53 PM

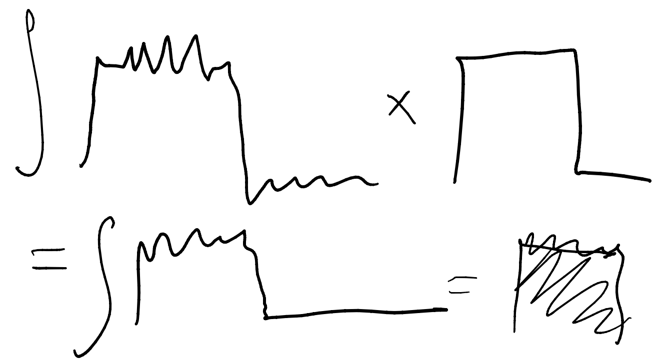
$p_1(t) =$  equal energy

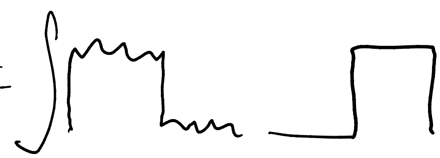
$p_2(t) =$ 

$p_1(t) + n(t) = r(t)$ 

CR:



$\langle r, p_1 \rangle = \int r(t)\, p_1(t)\, dt$ 

$\langle r, p_2 \rangle = \int r(t)\, p_2(t) =$ 

$= \int$  $dt =$  $\ll \langle r, p_1 \rangle$

Unless noise is massive, this should work equal energy, so = LS

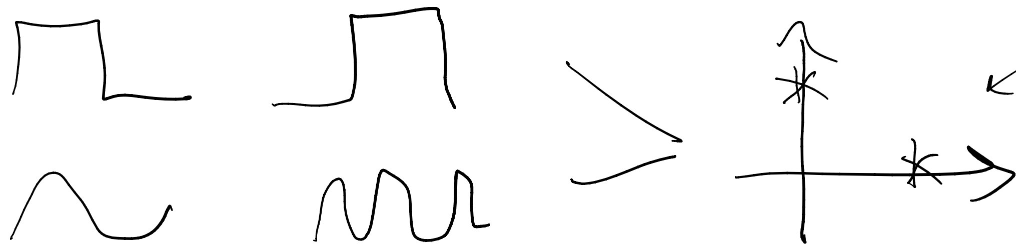
$$\|r-p_1\|^2 = \int |r-p_1|^2 \, dt \qquad \int \left(\quad - \quad \right)^2 dt = \int \left(\quad \right)^2 dt$$

$$\|r-p_2\|^2 = \int \left(\quad - \quad \right) dt = \int \left(\quad \right)^2$$ <u>huge</u>

Matched Filters

Tuesday, October 13, 2020 4:00 PM

from last week



only care about constellation

Say I have $\{s_i\}_{i=1}^{N}$ that I send, receive $r = s_i(t) + n(t)$ ← AWGN

now $r(t) \rightarrow \boxed{h(t)} \xrightarrow{y(t)}$ $\overset{T}{\diagup}$ — $r * h(T) = y(T)$

final signal $$y(T) = \int_0^T \left(s_i(t) + n(t)\right) h(T-t)\, dt$$

$$= \int_0^T s_i(t)\, h(T-t)\, dt + \int_0^T n(t) h(T-t)\, dt$$

signal + noise

def. $$SNR = \frac{\left(\int_0^T s_i(t)\, h(T-t)\, dt\right)^2}{\frac{N_0}{2} \int_0^T h^2(T-t)\, dt}$$ } just dep. on h, not s_i

filtered noise

$n \rightarrow \boxed{H} \rightarrow \frac{N_0}{2}|H|^2 = \text{power}$

$n \to \boxed{H} \to \frac{N_0}{2}|H|^2 = \text{power}$

say I want to find h which maximizes SNR

max. numerator $\left(\int_0^T s_i(t)\, h(T-t)\, dt\right)^2$

$$\left(\int_0^T f(t)\, g(t)\, dt\right)^2 \leq \int_0^T f(t)^2 dt \int_0^T g(t)^2\, dt$$ Cauchy-Schwartz

$$\langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2$$

here: $\left(\int_0^T s_i(t)\, h(T-t)\, dt\right)^2 \leq \int s_i^2(t)\, dt \int h^2(T-t)\, dt$

equality holds iff $\langle x, y\rangle = \|x\|\, \|y\|$

true iff $y = Cx$ for C constant

$$\langle x, Cx\rangle = C\langle x, x\rangle = C\|x\|^2$$
$$= \|x\|\, \|Cx\|$$

$\|x\| \; \|Cx\|$

So here, we have $h(T-t) = C s_i(t)$

or $h(T-t) \propto s_i(t)$

Matched filter $\boxed{h(T-t) = s_i(t)}$ maximizes SNR

$$SNR = \frac{\left(\int_0^T s_i(t)\, s_i(T-t)\, dt\right)^2}{\frac{N_0}{2}\int_0^T s_i(t)^2\, dt} = \frac{\mathcal{E}_s^2}{\frac{N_0}{2}\mathcal{E}_s} = \frac{2\mathcal{E}_s}{N_0}$$

intuitively $\mathcal{E}_s / (N_0/2)$ signal strength / noise strength

max SNR is good but in digital, really care about bit error rate (of course, related)

Compute a BER

Tuesday, October 13, 2020 4:18 PM

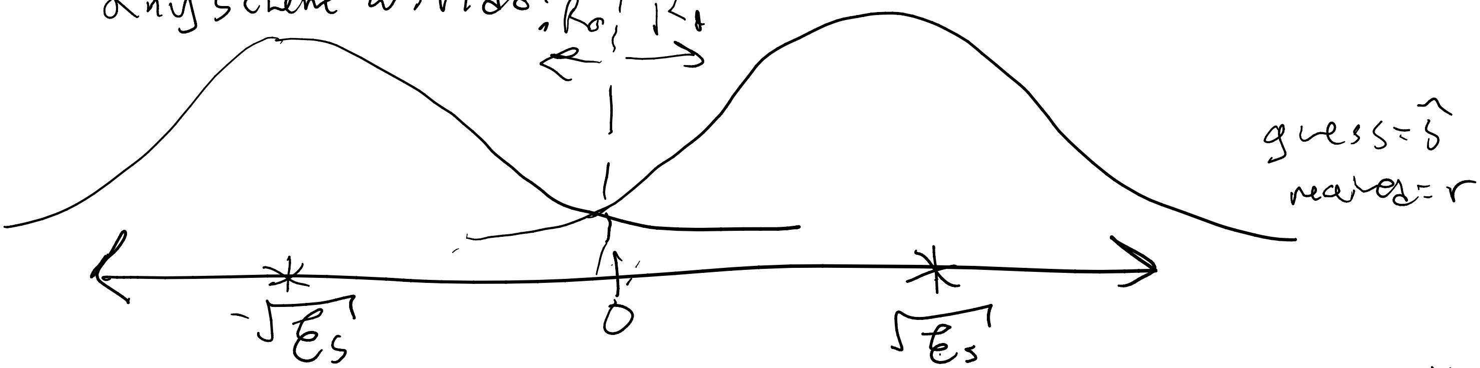
antipodal binary signalling, AWGN, equiprobable

equal energy, so $LS = CR = MF$ (max SNR)

AWGN, so $= ML$

equiprob, so $= MAP$ (min error prob.)

any scheme will do!



$$P[\text{error}] = P[\hat{s}=s_0 \mid s=s_1]\,\underbrace{P[s=s_1]}_{1/2} + P[\hat{s}=s_1 \mid s=s_0]\,\underbrace{P[s=s_0]}_{1/2}$$

$$= \frac{1}{2}\Big(P[\hat{s}=s_0 \mid s=s_1] + P[\hat{s}=s_1 \mid s=s_0]\Big)$$

$$r \mid s=s_1 \sim N\left(\sqrt{E_s}, \frac{N_0}{2}\right)$$

$$r \mid s=s_0 \sim N\left(-\sqrt{E_s}, N_0/2\right)$$

$$P[\hat{s}=s_0 \mid s=s_1] = P\left[N(\sqrt{E_s}, N_0/2) < 0\right]$$

$$P[\hat{s}=s_1 \mid s=s_0] = P\left[N(-\sqrt{E_s}, N_0/2) > 0\right]$$

$$P[\hat{s}=s_1 \mid s=s_0] = P[N(-\sqrt{\mathcal{E}_s}, N_0/2) > 0]$$

$$= P[\hat{s}=s_0 \mid s=s_1] \text{ by symmetry of Gaussian}$$

$$P[\text{error}] = \frac{1}{2}\left(2\, P[N(\sqrt{\mathcal{E}_s}, N_0/2) < 0]\right)$$

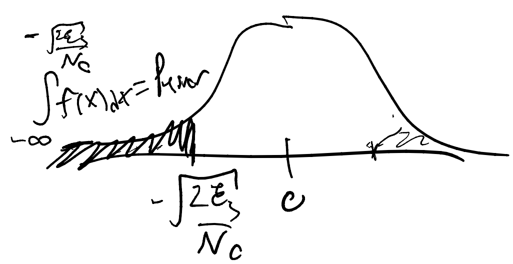
$$= P[N(\sqrt{\mathcal{E}_s}, N_0/2) < 0]$$

, say $Z \sim \mathcal{N}(0,1)$

$\rightarrow \sigma Z + \mu \sim N(\mu, \sigma^2)$

$$= P[N(0, N_0/2) < -\sqrt{\mathcal{E}_s}]$$

$$= P\left[Z < -\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$



def: $$Q(x) = \int_x^\infty f(y)\,dy$$ where $f(x)$ is std. normal pdf

"positive tail prob." opposite of CDF

$$P\left[Z < -\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right] \overset{\text{sym}}{=} P\left[Z > \sqrt{\frac{2\mathcal{E}_s}{N_0}}\right]$$

$$= Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)$$

so $$\boxed{P[\text{error}] = Q\left(\sqrt{\frac{2\mathcal{E}_s}{N_0}}\right)}$$

antipodal
AWGN
equiprobable

in MATLAB - Q = qfunc (its inverse is there too)

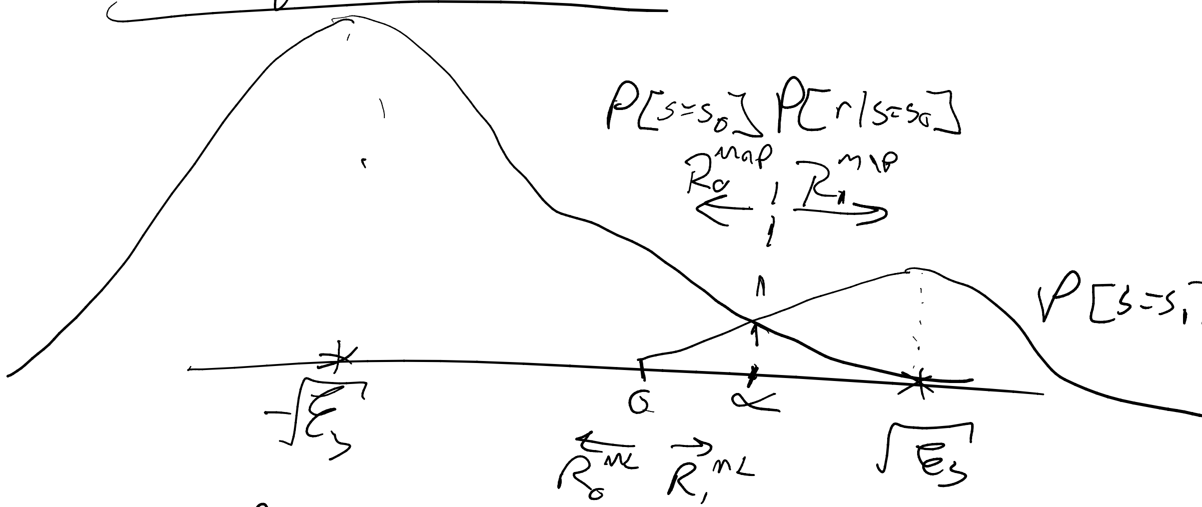
as $N_0 \nearrow$ $\sqrt{\frac{2E_s}{N}} \to 0$ $Q(0) = 1/2$ "guessing"

as $E_s \nearrow$ $\sqrt{\frac{2E_s}{N}} \nearrow \infty$ $Q(\infty) = 0$ Perfect

Finding MAP regions

Tuesday, October 13, 2020 4:38 PM

not-equiprobable case



$P[s=s_0] > P[s=s_1]$

$P_0 > P_1$

using MAP

$$P[\text{error}] = P[s=s_1]P[\hat{s}=s_0|s=s_1] + P[s=s_0]P[\hat{s}=s_1|s=s_0]$$

$r|s=s_1 \sim N(\sqrt{E_s}, N_0/2)$, $\quad P[\hat{s}=s_0|s=s_1] = P[N(\sqrt{E_s}, \frac{N_0}{2}) < \alpha]$

$r|s=0 \sim N(-\sqrt{E_s}, N_0/2)$, $\quad P[\hat{s}=s_1|s=s_0] = P[N(-\sqrt{E_s}, \frac{N_0}{2}) > \alpha]$

$$P_{error} = P_1 \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\alpha} e^{-(x-\sqrt{E_s})^2/N_0} dx + P_0 \frac{1}{\sqrt{\pi N_0}} \int_{\alpha}^{\infty} e^{-(x+\sqrt{E_s})^2/N_0} dx$$

↑ true for arbitrary α

I want to find <u>best</u> α (I know its at the overlap, but I want a closed form)

find optimal α: $\frac{\partial}{\partial \alpha} P_{error} = 0$

$$0 = \cancel{\frac{1}{\sqrt{\pi N_0}}} \left(P_1 e^{-(\alpha-\sqrt{E_s})^2/N_0} - P_0 e^{-(\alpha+\sqrt{E_s})^2/N_0} \right)$$

$$0 = \cancel{\frac{1}{\sqrt{\pi N_0}}}\left(p_1 e^{\cdots/N_0} - p_0 e^{\cdots}\right)$$

$$\frac{p_0}{p_1} = e^{-(\alpha-\sqrt{E_s})^2/N_0}\, e^{(\alpha+\sqrt{E_s})^2/N_0}$$

$$= \exp\left(4\alpha\sqrt{E_s}/N_0\right)$$

optimal α (i.e. where the overlap occurs) is

$$\boxed{\alpha = \frac{N_0}{4\sqrt{E_s}} \log\left(p_0/p_1\right)}$$

Sanity check: if $p_0 = p_1$, $\log(p_0/p_1) = 0$

= ML decision region

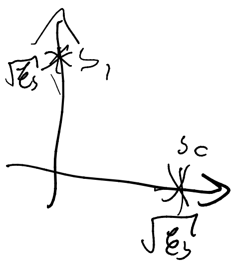
$p_0 > p_1 \Rightarrow \alpha > 0$

$p_0 < p_1 \Rightarrow \alpha < 0$

P_{error} = (look above and plug in this α, if you dare)

Orthogonal

Tuesday, October 13, 2020 5:01 PM



equiprobable, AWGN $\quad ML = LS = MAP = CR$

$$y_0(T) = \int_0^T r(t)\, s_0(T-t)\,dt, \quad y_1(t) = \int_0^T r(t)\, s_1(T-t)\,dt$$

$$P[\text{error}] = \frac{1}{2} P[\hat{s} = s_0 \mid s = s_1] + \frac{1}{2} P[\hat{s} = s_1 \mid s = s_0]$$

CR: $\max_i y_i$ is the signal sent

$$P[\text{error} \mid s = s_1] \stackrel{\text{CR}}{=} P[y_0(T) > y_1(T) \mid s = s_1]$$

$$P[\text{error} \mid s = s_0] = P[y_1(T) > y_0(T) \mid s = s_0]$$

$$P[\text{error}] = \frac{1}{2}\left(P[y_0 > y_1 \mid s = s_1] + P[y > y_0 \mid s = s_0]\right)$$

$z_1 = y_1 - y_0$ given s_1 was sent, $y_1 = \sqrt{\mathcal{E}_s} + n_1$ ← filtered noise
$y_0 = n_0$

$$= \sqrt{\mathcal{E}_s} + n_1 - n_0$$

$$z_0 = y_0 - y_1$$
$= \sqrt{\mathcal{E}_s} + n_0 - n_1$, given s_0 was sent

n_0, n_1 0-mean indep Normal $\sigma^2 = N_0/2$ so

$n_1 - n_0$ and $n_0 - n_1 \sim N(0, N_0)$

↑ not $N_0/2$

CLT without the factor of ½
(Chapter 5 of Hogg Tanis)

(Chapter 5 of Hogg Tanis)

$$Z_1 = N(\sqrt{\mathcal{E}_s}, N_0) = Z_2$$ noise process seen by corr. receiver

$$P[y_0 > y_1 \mid s = s_1] = P[y_1 - y_0 < 0 \mid s = s_1]$$
$$= P[Z_1 < 0]$$
$$P[y_1 > y_0 \mid s = s_0] = P[Z_1 < 0]$$

$$P_{error} = \frac{1}{2}\left(P[Z_1 < 0] \cdot 2\right) = P[N(\sqrt{\mathcal{E}_s}, N_0) < 0]$$
$$= P\left[\underset{\substack{\uparrow \\ \text{std normal}}}{Z} < \sqrt{\mathcal{E}_s / N_0}\right]$$

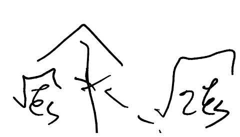
$$\boxed{P_{error} = Q\left(\sqrt{\mathcal{E}_s / N_0}\right)}$$

worse than antipodal

does that make sense?

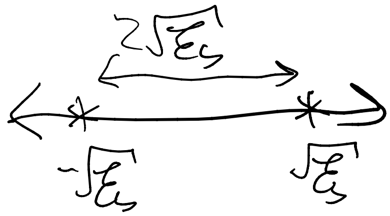
yes!

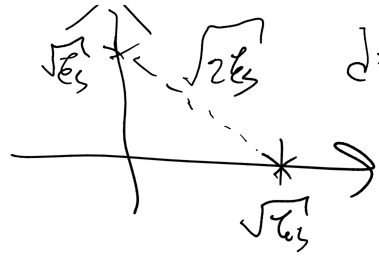
$2\sqrt{\mathcal{E}_s}$



distance is much less

yes:





distance is much less (factor of $\sqrt{2}$)

Note: factor of root 2 appears inside qfunc, nonlinear (very)

For a given SNR/bit $= \dfrac{\mathcal{E}_b}{N_0}$, here $\mathcal{E}_b = \mathcal{E}_s$ (2 symbols, 1 bit/symbol)

orthogonal worse than antipodal

in binary case, orthogonal used very rarely

learn it here b/c generalizes to high-dim

Generally, for binary schemes we learned,

let $d_{01} = \|s_0 - s_1\|$, then

AWGN equiv. →

$$\boxed{P_{error} = Q\left(\sqrt{\frac{d_{01}^2}{2N_0}}\right)}$$

vector-encoded

Tuesday, October 13, 2020 5:21 PM

multiple symbols: $2^k = M$ symbols

each symbol represents k bits of data

"M-ary signalling"

Say I send $R_s = 1/T$ symbols/second

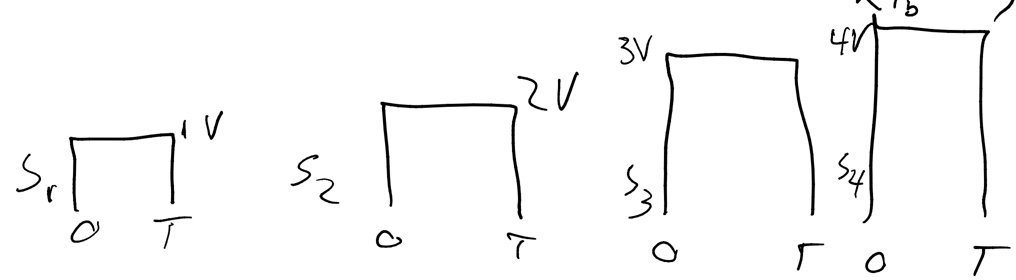
$\rightarrow R_b = k/T$ bits/second

$\nearrow M$ increases my bitrate if T stays the same

M-ary signaling allows for much faster transmission than binary

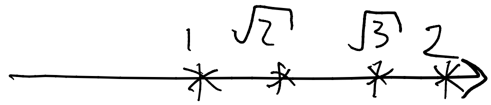
Def. Bit interval $T_b = 1/R_b = T/k$ (you don't actually "send a bit" in T_b, you send k bits in kT_b)

Simple example:

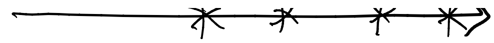


$s_1 \rightarrow 00$, $s_2 \rightarrow 01$, $s_3 \rightarrow 11$, $s_4 \rightarrow 10$

$\psi = \sqrt{1/T}$



4-ary pulse-amplitude



Pulse-amplitude modulation (PAM)

should be clear — further away points are, the less noise matters

Want many points (higher M), to spread more I need higher energy

M-ary orthogonal

Tuesday, October 13, 2020 5:30 PM

$$s_i(t) = A\cos(2\pi i w_0 t), \quad i = 1, 2, \ldots, M, \quad 0 \leq t \leq T_s$$

$$s_i \perp s_j \quad \forall i \neq j$$

lives in an M-dimensional space

"M-ary FSK" (freq-shift keying)

Tuesday, October 13, 2020 5:32 PM

effect of noise on M-ary schemes (AWGN)

$r(t) = s_i(t) + n(t)$, $\sigma^2 = N_0/2$, $\mu = 0$

if equiprobable, equal energy, $LS = \underset{(CR)}{MF} = ML = MAP$

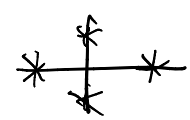
M-ary MF

suppose $\{\psi_j\}_{j=1}^{N}$ are an o.n.b. for $\{s_i\}_{i=1}^{M}$

N-dim signal space, const. contains M symbols

$$s_i(t) = \sum_{k=1}^{N} s_{ik}\,\psi_k(t)\ ,\ 0 \le t \le T_s$$

ex. 4-star I/Q constellation



$A\cos\omega_c t = s_1$

$-A\cos\omega_c t = s_2$ $\quad M=4$

$-A\sin\omega_c t = s_3$ $\quad N=2$

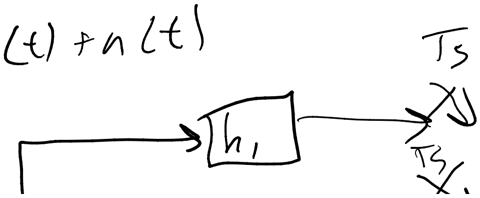
$A\sin\omega_c t = s_4$

can write s_i as a vector in $\mathbb{R}^n$ $\begin{pmatrix} s_{i1} \\ \vdots \\ s_{iN} \end{pmatrix}$

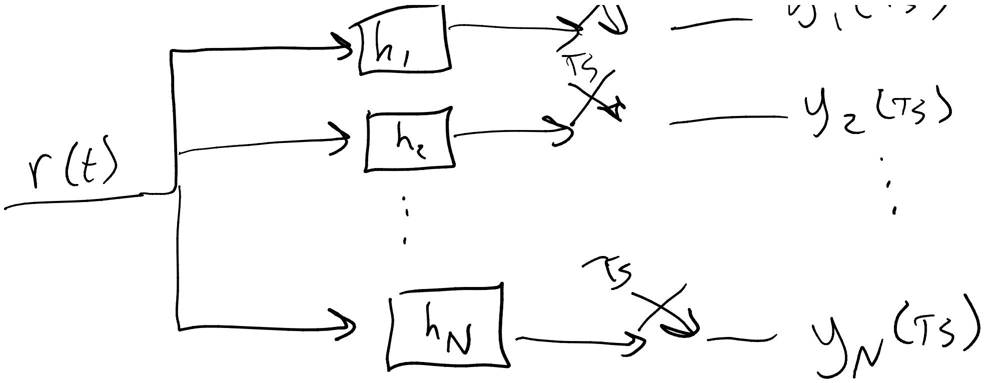
say $h_i(t) = \psi_i(T_s - t)$

then $s_i(t) * h_j(t) = \displaystyle\int_0^{T_s} s_i(t)\,\psi_j(T_s - t)\,dt = s_{ij}$

$r(t) = s_i(t) + n(t)$



— $y_1(T_s)$



$$y_j(T_s) = \int_0^{T_s} r(t)\,\psi_j(T_s - t)\,dt$$

$$= \int_0^{T_s} s_i(t)\,\psi_j(T_s-t)\,dt + \overbrace{\int_0^{T_s} n(t)\,\psi_j(T_s-t)\,dt}^{\text{filtered noise}}$$

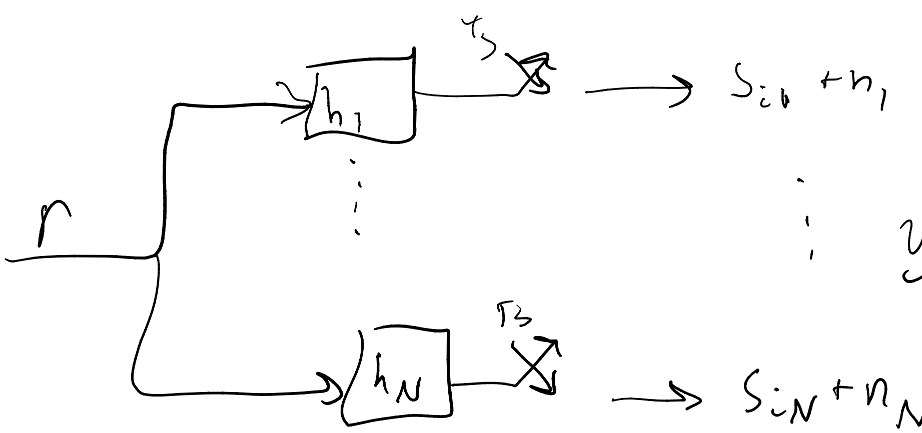
$$= n_j$$

$$= \int_0^{T_s} \sum_{k=1}^{N} s_{ik}\psi_k(t)\,\psi_j(T_s-t)\,dt + n_j$$

$\psi_k \perp \psi_j$ if $j \neq k$

$\langle \psi_k, \psi_j \rangle = 1$ if $j = k$

$$\boxed{y_j = s_{ij} + n_j}$$



$$y = \begin{pmatrix} s_{i1} \\ \vdots \\ s_{iN} \end{pmatrix} + \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}$$

$$= s_i + n$$

$\longrightarrow \boxed{h_N} \longrightarrow \otimes \longrightarrow s_{iN} + n_N$

$= s_i + n$

proj. of noise onto N-dim signal space

What is n like?

$$\mathbb{E}[n_i] = \int_0^{T_s} \mathbb{E}[n(t)]\,\psi_i(t)\,dt = 0, \quad n_i \text{ 0-mean}$$

Cov.

$$\mathbb{E}[n_i n_j] = \int_0^{T_s}\int_0^{T_s} \mathbb{E}[n(t)\,n(\tau)]\,\psi_i(t)\psi_j(\tau)\,dt\,d\tau$$

white so autocorr is $\sigma^2\delta(t-\tau)$

$$= \frac{N_0}{2}\int_0^{T_s}\int_0^{T_s} \delta(t-\tau)\,\psi_i(t)\psi_j(\tau)\,dt\,d\tau$$

$$= \frac{N_0}{2}\int_0^{T_s} \psi_i(\tau)\psi_j(\tau)\,d\tau$$

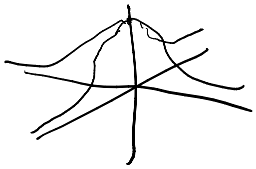
$$= \begin{cases} \frac{N_0}{2} & i=j \\ 0, & i\neq j \end{cases}$$

$$n \sim N\left(\vec{0}, \frac{N_0}{2} I\right) \qquad \frac{N_0}{2} I = \Sigma$$

PDF of $N(\vec{0}, \frac{N_0}{2} I)$

$$f_n(x) = \prod_{i=1}^{N} f_{n_i}(x_i) = \frac{1}{(\pi N_0)^{N/2}} e^{-\sum_{i=1}^{N} x_i^2 / N_0}$$

in 2-D



isotropic centered at 0 in any dimension