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X WSS, H LTI

Even fcn

then $S_X = \mathcal{F}\{R_X\}$ $\longleftrightarrow$ nonneg. real

$$X \to \boxed{H} \to Y$$

then X, Y jointly WSS

$$S_Y = |H|^2 S_X, \quad \mu_Y = H(0)\,\mu_X$$

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Def. X, Y jointly WSS, we define

$$S_{XY}(\omega) = \mathcal{F}\{R_{XY}\}$$

"Cross-Spectral Density"

Gaussian Processes

Def random variables $X_1, \ldots, X_n$ are jointly Gaussian if every linear comb. of X_is is Gaussian

⊛ special case: $X_1, \ldots, X_n$ independent, identical Gaussian

Def a random process $X(t)$ is a Gaussian process if $\forall n \in \mathbb{N}$, $\forall (t_1, \ldots, t_n) \in \mathbb{R}^n$ $\{X(t_1), X(t_2), \ldots, X(t_n)\}$ are jointly gaussian.

$\boxed{\text{⊛ special case indep. gaussian identical at all } t \in \mathbb{R}}$

Def Jointly Gaussian Processes $X(t)$, $Y(t)$ j.g. if $\forall n, m \in \mathbb{N}$ $\forall (t_1, \ldots, t_n, \tau_1, \ldots, \tau_m) \in \mathbb{R}^{n+m}$, $\{X(t_1), \ldots, Y(\tau_m)\}$ j.g. as r.v's

$\forall (t_1, \dots, t_n, \tau_1, \dots, \tau_m) \in \mathbb{R}^{\cdots}$, $\{X(t_1), \dots, Y(\tau_m)\}$ j.g. as r.v.s

1) If $X(t)$ Gaussian, H LTI

$$X \to \boxed{H} \to Y$$

$X(t)$, $Y(t)$ are jointly gaussian

2) For j.g. processes

uncorrelated = independent

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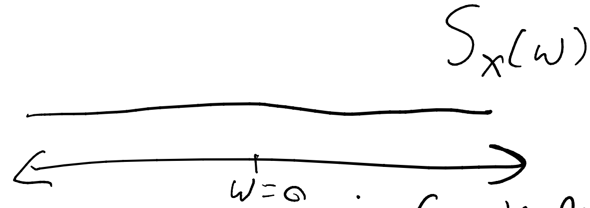
Def. A process is <u>white</u> if it has flat PSD

i.e. $S_X(w)$ is constant fncn



<u>infinite power</u>

impossible

$$P_X = \infty$$

usually, when we say "white" in practice, we mean flat PSD over the band of interest

if $\quad S_X(w) = C \;\; \forall w$, then

$$\mathcal{F}^{-1}\{C\} = R_X(\tau) = C\,\delta(\tau)$$

autocorr. of a white process is δ, i.e. no two time indices are correlated with eachother

Ex. $X(t)$ gaussian process, uncorr = indep.

So indep. identical gaussian at each $t \Rightarrow$ white

↑ most noise in this class is modeled

↑ most noise in this class is modeled this way

A **W** **G** **N**

Additive White Gaussian Noise

Signal $m(t)$

model for AWGN is

$$\boxed{r(t) = m(t) + n(t)}$$

↑ r.p. white gaussian

usually means indep. identical gaussian $\forall t$

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Independent $N(0, \sigma^2)$ at all time

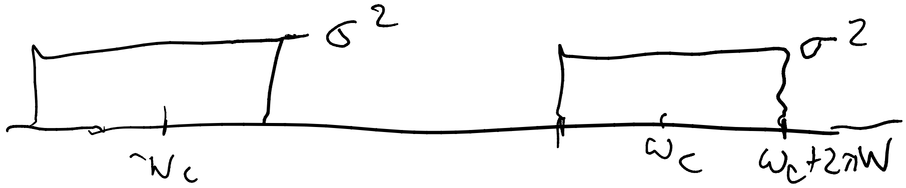
$$R(\tau) = 0 \ \forall \ \tau \neq 0$$

$$R(0) = E[X^2] = \sigma^2$$

$$\mathcal{F}\{R(\tau)\} = \mathcal{F}\{\sigma^2 \delta(t-t_0)\} = \boxed{\sigma^2 = S_X(\omega)}$$

We will have a receiver which will filter all incoming signals to BW of interest

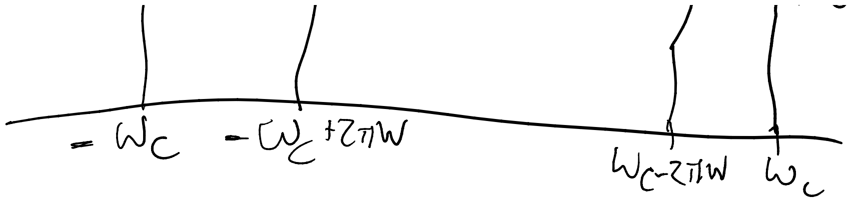
So filtered noise looks like



PSD of the filtered noise

by convention, we say $\boxed{\sigma^2 = N_0/2}$ ← noise power

S_N



$$P_N = \frac{1}{2\pi}\int S_N(\omega)\,d\omega = \frac{N_0}{2}\frac{(2\pi W)2}{2\pi} = N_0 W$$

power in SSB noise is $N_0 W = 2\sigma^2 W$

power in DSB noise is $2N_0 W = 4\sigma^2 W$

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Noise on Baseband Signals

Model noise as white $S_N(\omega) = N_0/2$

Channel model

$\text{in} \longrightarrow \oplus \longrightarrow \text{output}$

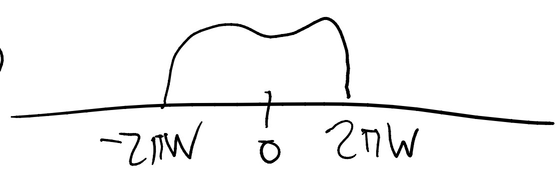
$\uparrow$

noise

AWN model

send bb signal

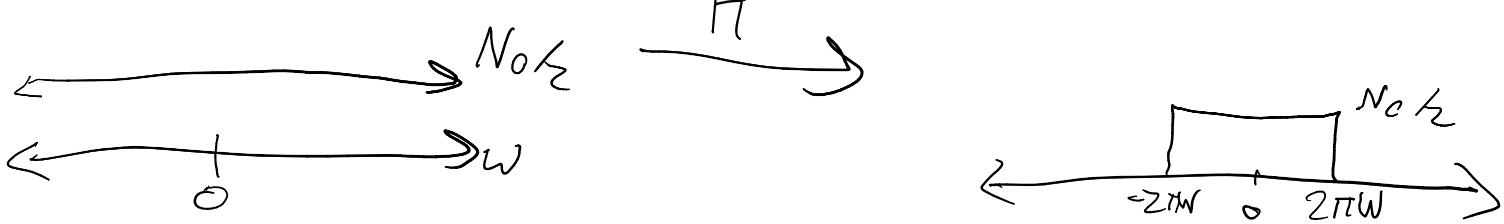
$X(\omega)$



at the receiver, we apply BPF $H(\omega) = \begin{cases} 1, & |\omega| \le 2\pi W \\ 0, & \text{else} \end{cases}$

$S_N(\omega)$

$|H(\omega)|^2 S_N(\omega) = S_n(\omega)$



$X(\omega) \xrightarrow{\quad H \quad} X(\omega)$ b/c $X(\omega) = 0$ for $|\omega| \ge 2\pi W$

$X(\omega)$ ⟶ $X(\omega)$ b/c $X(\omega)=0$ for $|\omega|\geq 2\pi W$

$P_N = \infty$ white noise has infinite power

$$P_n = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} \frac{N_0}{2}\,d\omega = \frac{4\pi W}{2\pi}\,\frac{N_0}{2} = N_0 W$$ ← power in baseband filtered noise

(if AWGN, $N_0 = 2\sigma^2$ so $P_n = 2\sigma^2 W$)

Obvious but important: additive → P_n is indep. of signal power

... Sort of...

P_n is a fncn of W, so $W \nearrow$ (is a prop. of the signal)
$P_n \nearrow$

first example of the power-bw relationship (tradeoff)

Signal-to-Noise ratio (SNR) is the ratio of received signal power over received noise power (usually reported in dB)

in our case $$SNR_{bb} = \frac{P_r}{N_0 W}$$ ← power of received message signal in our case $SNR_{bb} = \frac{\text{[illegible]}}{N_0 W}$

$$SNR_{dB} = \underbrace{10}\log_{10}\left(\frac{P_r}{P_N}\right)$$

10 b/c it's power

Ex. AWGN, variance 5×10^{-12}
afflicts bb signal w/ bw 10 kHz
I transmit w/ 100 kW of power
but the channel attenuates by a factor of 10^{-10}

$$P_R = (100\text{kW})(10^{-10}) = 10^{-5}\,W$$

$$P_N = N_0 W = (10\text{kHz})(\sigma^2 2) = (10^4\,\text{Hz})(10^{-11})$$
$$= 10^{-7}\,W$$

linear scale $\rightarrow$ $SNR = \frac{P_R}{P_N} = \frac{10^{-5}}{10^{-7}} = 100$

log scale $\rightarrow$ $SNR_{dB} = 10\log_{10}(100) = 20\,dB$

Noise in AM

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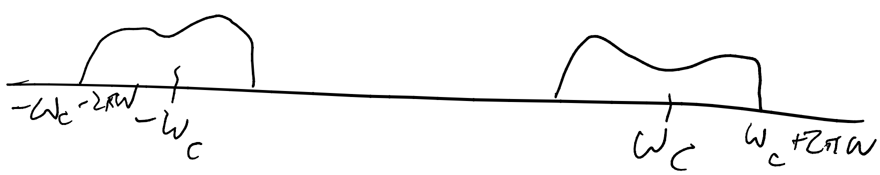
DSB-SC

we transmit $u(t) = A_c m(t) \cos \omega_c t$

$|U(\omega)|$



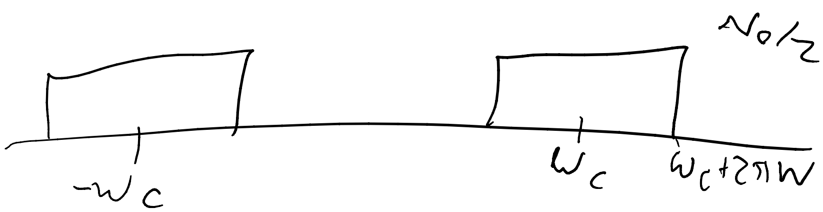
model: received = signal + AWN

$$r(t) = u(t) + n(t)$$

filtered white noise
$\omega_c - 2\pi W \le |\omega| \le \omega_c + 2\pi W$

Sidetrack - talk about filtered white noise that isn't at bb

$S_N(\omega)$



just like w/ deterministic signals, we can write a bb rep. of this noise:

$$Y(t) = I_n(t) \cos \omega_c t - Q_n(t) \sin \omega_c t$$

random IQ processes

It can be shown that if X is white, Gaussian

1) I_n, Q_n are zero-mean, baseband, jointly WSS and jointly gaussian

2) $P_Y = P_{I_n} = P_{Q_n} \left(= \frac{1}{2\pi} \int S_Y(\omega)\, d\omega \right)$

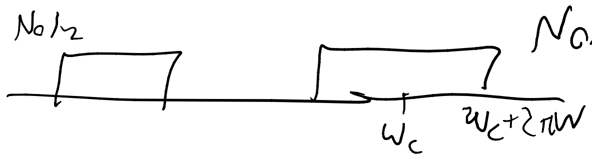
3) I_n and Q_n have same PSD

So $P_{Q_n} = P_{I_n}$
$= P_Y = 2 N_0 W$

4) if ω_c is an axis of symmetry (i.e. in DSB) then I_n, Q_n are indep.

$u(t) = A_c m(t) \cos \omega_c t$

filtered noise



AWGN

Can write $r(t) = u(t) + n(t)$

$= u(t) + n_I(t) \cos \omega_c t - n_Q(t) \sin \omega_c t$

Then we demodulate $r(t)$ by product w/ $\cos(\omega_c t + \varphi)$ then baseband filter

$\cos a \cos b = \frac{1}{2} \cos(a+b) + \frac{1}{2} \cos(a-b)$

$\cos a \sin b = \frac{1}{2} (\sin(a+b) - \sin(a-b))$

$\cos a \sin b = \frac{1}{2}(\sin(a+b) - \sin(a-b))$

$$r(t)\cos w_c t = A_c m(t)\cos w_c t \cos(w_c t + \varphi) + n_I(t)\cos w_c t \cos(w_c t + \varphi) - n_Q(t)\sin w_c t \cos(w_c t + \varphi)$$

$\downarrow$ filter plus simplify

$$y(t) = \underbrace{\frac{1}{2}A_c m(t)\cos\varphi}_{\text{What we had w/ no noise}} + \underbrace{\frac{1}{2}\left(n_I(t)\cos\varphi + n_Q(t)\sin\varphi\right)}_{+\text{ bb noise}}$$

$\uparrow$ this is stopping pt for non-coherent

assume coherent, then $\varphi = 0$

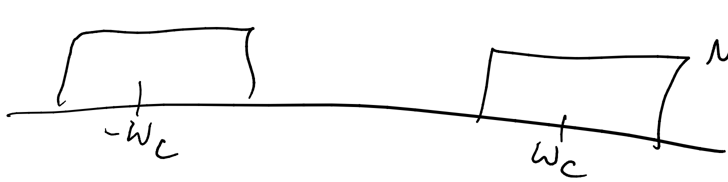
$$y(t) = \underbrace{\frac{1}{2}A_c m(t)}_{\text{What we had w/ no noise for coh.}} + \frac{1}{2}\underbrace{n_I(t)}_{P_{n_I} = P_N}$$

by point (2)

So power in signal (as before) is $P_o = \frac{A_c^2}{4} P_M$

power in noise is $\frac{1}{4}P_N = P_{no}$

$S_N(\omega)$



$\rightarrow P_N = \frac{N_0}{2}\frac{1}{2\pi}(8\pi W)$
$= 2N_0 W$

So power in noise $P_{no} = \frac{N_0 W}{-} = \sigma^2 W$

So power in noise $P_{n_o} = \frac{N_o W}{2} = \sigma^2 W$

$$SNR_{\substack{DSBSC\\ \text{Coherent}}} = \frac{A_c^2/4\, P_M}{N_o W/2} = \boxed{\frac{A_c^2}{2N_oW} P_M}$$

$$= \frac{A_c^2}{4\sigma^2 W} P_M$$

SNR_{bb} was $\frac{P_R}{N_o W}$, $\quad P_R = \frac{1}{T}\int_0^T \left(A_c \cos w_c t\, m(t)\right)^2 dt$

$$= \frac{A_c^2 P_m}{2}$$

$$\boxed{SNR_{bb} = \frac{P_R}{N_o W} = \frac{A_c^2 P_M}{2N_o W} = SNR_{DSB\text{-}SC}}$$

coherent DSB-SC has same SNR as BB

<u>SSB</u> $\quad r(t) = u(t) + n(t)$ AWGN

$$u(t) = A_c m(t) \cos w_c t \mp A_c \hat{m}(t) \sin w_c t$$

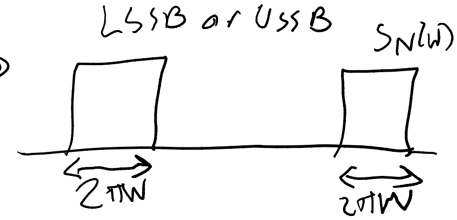
USSB ($-$), LSSB ($+$)

$$r(t) = \left(A_c m(t) + n_I(t)\right)\cos w_c t - \left(\pm A_c \hat{m}(t) + n_Q(t)\right)\sin w_c t$$

$\downarrow$ demod + BB filter, assume coherent

$$y(t) = \frac{A_c}{2} m(t) + \frac{1}{2} n_I(t) \qquad (\text{for USSB or LSSB})$$

$$= P_o + P_{n_c} = \frac{A_c^2}{4} P_M + \frac{1}{4} P_n$$



$$P_n = \frac{1}{2\pi} \frac{N_o}{2} 4\pi W = N_o W$$

$$SNR_{SSB} = \frac{\frac{1}{4} A_c^2 P_M}{\frac{1}{4} N_o W} = \boxed{\frac{A_c^2 P_M}{N_o W}}$$

Seems better than DSB

but we transmitted $m(t)$ and $\hat{m}(t)$ at same power

So we receive 2 times signal power

so if $P_R^{DSB} = \frac{A_c^2}{2} P_M$, then $P_R^{SSB} = A_c^2 P_M$

so

$$\boxed{SNR_{SSB} = \frac{P_R}{N_o W} = SNR_{bb} = SNR_{DSB\text{-}SC}}$$

P_R is more directly related to transmitted power ← true cost

whereas P_{signal} in SNR computation is demod signal power

SSB - transmit 2* power, half BW, same SNR

power/BW tradeoff

Conventional

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$$u(t) = A_c(1 + a\,m(t))\cos w_c t$$

$$r(t) = \left(A_c(1 + a\,m(t)) + n_I(t)\right)\cos w_c t - n_Q(t)\sin w_c t$$

before if I $\otimes \cos w_c t$,
I get rid of $n_Q(t)$ by cos ⊥ sin

here, demod is rectifier + LPF → doesn't get rid of n_Q

Using conventional demod → quadrature component noise
→ Worse than DSB or SSB

What if I have conventional AM and demod like regular?
= Like DSB AM, with a pilot tone

↓

$$y_1(t) = \underbrace{\frac{1}{2}A_c(1}_{DC} + a\,m(t)) + \frac{1}{2}n_I(t)$$

↓ use a DC-block (capacitor)

$$y(t) = \frac{aA_c}{2}m(t) + \frac{1}{2}n_I(t)$$

$$P_N = P_N^{DSB} = \frac{2N_0W}{4}, \quad P_R = \frac{A_c^2}{2}(a^2P_M + 1)$$

$$P_o = \frac{a^2A_c^2P_M}{4}$$

$$P_o = \frac{a^2 A_c^2 P_M}{4}$$

So $SNR_{DSB-conv} = \frac{a^2 A_c^2 P_M}{2 N_0 W} = \frac{a^2 P_M}{1+a^2 P_M} \quad \frac{\frac{A_c^2}{2}(a^2 P_M + 1)}{N_0 W}$

$$= \eta \frac{P_R}{N_0 W} = \eta \, SNR_{bb}$$

$$\eta = \frac{a^2 P_M}{1+a^2 P_M} < 1$$

So With a pilot tone, SNR decreased by η

waste power in pilot tone

Useless carrier component $\rightarrow$ large part of P_R

FM/PM

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$$A \cos(\omega_c t + \theta)$$

Modulating A → AM

modulating ω_c → FM "frequency modulation"

Modulating Phase → PM "phase modulation"

"Angle modulation"

Big Problem - not intuitive at all
- nonlinear - modulating the argument of a transcendental function

We have to rely on hard math more than intuition

Generally: write any angle-modulated signal as

$$u(t) = A_c \cos(\omega_c t + \varphi(t))$$

how will this affect frequency content of the signal?

One simple example: let $\varphi(t) = \omega_0 t + \theta$ - general linear phase

$$u(t) = A_c \cos((\omega_c + \omega_0)t + \theta)$$ increased freq

but still narrowband

what if $\varphi(t) = \sin \omega_0 t$? - We have no tools to what if $\varphi(t) = \sin \omega_0 t$? — We have no tools to deal with this yet

Def. The instantaneous frequency of an angle-modulated signal $u(t) = A_c \cos(\omega_c t + \varphi(t))$ is

$$f_i(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \varphi(t)$$

or

$$\omega_i(t) = \omega_c + \frac{d}{dt} \varphi(t) \quad \text{in radian freq.}$$

in above example, $\varphi(t) = \omega_0 t + \theta$, $\varphi'(t) = \omega_0$

$\omega_i = \omega_c + \omega_0$ = freq. of mod. signal

"More-or-less" freq. of signal at time t

in PM

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We Let $\boxed{\varphi(t) = k_p m(t)}$ → phase ∝ message

in FM

we let the instantaneous freq. deviation is prop. to the message $f_i - f_c \propto m$

$$f_i(t) - f_c = k_f m(t) = \frac{1}{2\pi} \frac{d}{dt} \varphi(t)$$

in FM $\varphi \propto \int m(t)\, dt$

Specifically $$\boxed{\varphi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)\, d\tau}$$

FM might better be called "instantaneous freq. mod"

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How do we demod?

we want to find $\varphi(t)$ somehow

ex. Let $m(t) = a\cos\omega_0 t$

$$\text{PM: } \varphi(t) = aK_p \cos\omega_0 t$$

$$\text{FM: } \varphi(t) = 2\pi a K_f \int_{-\infty}^{t} \cos\omega_0 \tau \, d\tau$$

$$= \frac{2\pi a K_f}{\omega_0} \sin\omega_0 t$$

$$u_{PM}(t) = A_c \cos\left(\omega_c t + aK_p \cos\omega_0 t\right)$$

$$u_{FM}(t) = A_c \cos\left(\omega_c t + \frac{2\pi a K_f}{\omega_0} \sin\omega_0 t\right)$$

$$\text{def } \tilde{\beta}_p = aK_p, \quad \tilde{\beta}_f = \frac{aK_f}{f_0}$$

$$u_{PM}(t) = A_c \cos\left(\omega_c t + \tilde{\beta}_p \cos\omega_0 t\right)$$

$$u_{FM}(t) = A_c \cos\left(\omega_c t + \tilde{\beta}_f \sin\omega_0 t\right)$$

PM: Phase changes by at most $\tilde{\beta}_p$

More generally: $K_p \max(|m(t)|) = \Delta\varphi_{max}$

FM: $K_f \max(|m(t)|)$ ...

max

FM: $\frac{k_f \max(|m(t)|)}{W} = \Delta\varphi_{max}$, W BW of $M(\omega)$

Def Modulation index

PM: $\beta_p = k_p \max(|m(t)|) = \Delta\varphi_{max}$

FM: $\beta_f = \frac{k_f \max(|m(t)|)}{W} = \frac{\Delta f_{max}}{W}$

Special case $\varphi(t) \ll 1 \ \forall t$ (low mod. index)

"low index" or "Narrowband" FM/PM

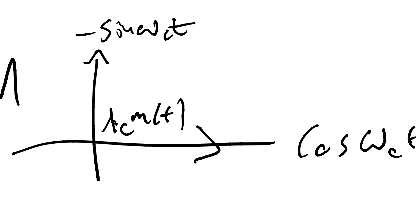
$u(t) = A_c \cos(\omega_c t + \varphi(t))$

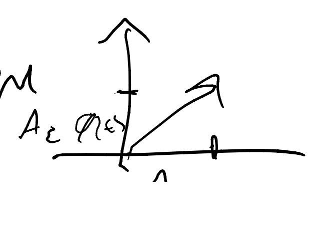
$= A_c \cos(\omega_c t)\cos(\varphi(t)) - A_c \sin(\omega_c t)\sin(\varphi(t))$

$\approx \underbrace{A_c \cos \omega_c t}_{\cos x \approx 1} - \underbrace{A_c\, \varphi(t) \sin \omega_c t}_{\sin x \approx x}$

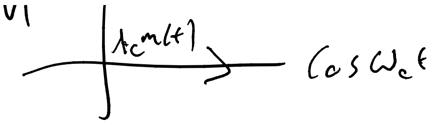
for small x

$I(t) = A_c$, $Q(t) = A_c \varphi(t)$

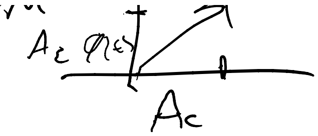
in AM 

in FM/PM 

in AM



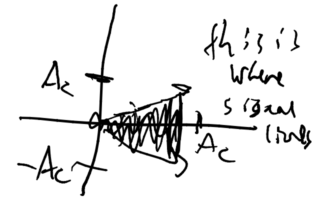
in FM/PM



narrowband $\rightarrow I(t) = A_c \quad Q(t) = A_c\varphi(t)$

More generally $\rightarrow I(t) = A_c\cos\varphi(t)\,,\ Q(t) = A_c\sin\varphi(t)$

range $[-A_c, A_c]$



In the freq. domain

Special case $\quad m(t) = a\cos\omega_0 t$

$$u_{FM}(t) = A_c\cos(\omega_c t + \beta_f \sin\omega_0 t)$$

$$= Re\left(A_c e^{j\omega_c t} e^{j\beta_f \sin\omega_0 t}\right)$$

periodic with period $T = \frac{2\pi}{\omega_0}$

So we can rep. as a Fourier Series

$$e^{j\beta_f \sin\omega_0 t} = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 t n}$$

where

$$c_n = \frac{1}{T}\int_0^T e^{j\beta_f \sin\omega_0 t} e^{-jn\omega_0 t}\,dt$$

def. of F.S.

$$\frac{1}{2\pi}\int^{2\pi} e^{j(\beta\sin u - nu)}$$

$$= \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta \sin u - nu)}\, du}, \quad u = \omega_0 t$$

evaluate this integral X it's impossible

This is an indexed set of functions of β with a name!

<u>Def.</u> The <u>n^{th} order Bessel Function of the first kind</u>
for integer n $\quad J_n : \mathbb{R} \to \mathbb{R}$ is def as

$$J_n(\beta) = \frac{1}{2\pi}\int_0^{2\pi} e^{j(\beta \sin u - nu)}\, du$$

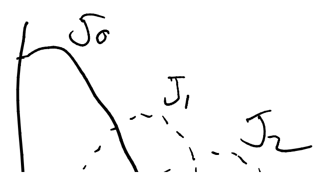
<u>Shockingly useful</u>

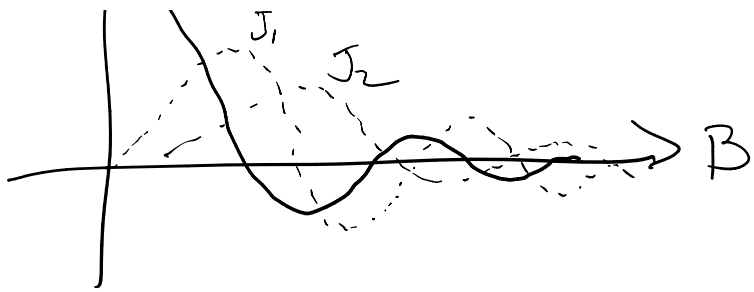
We have a <u>lot</u> of info about Bessel Fncns

1) $J_{-n} = (-1)^n J_n$, so $|J_n(\beta)| = |J_{-n}(\beta)|$

2) $\max\limits_{\beta>0}\left(J_n(\beta)\right) > \max\limits_{\beta>0}\left(J_{n+1}(\beta)\right) \;\forall n$

3) I will draw them for you <u>poorly</u>





4) $J_n(\beta) = \sum_{k=0}^{\infty} \frac{(-1)^k (\beta/2)^{n+2k}}{k!\,(n+k)!}$

for small β, $J_n(\beta) \approx \frac{\beta^n}{2^n\, n!}$ so as $n \nearrow$ $J_n(\beta) \searrow$ for small β

For higher β, you need more modes to estimate power

for $\beta \leq 8$ can get 98% power est. with <10 terms of sum

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$$u(t) = Re\left(A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j\omega_0 n t} e^{j\omega_c t}\right)$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos((\omega_c + n\omega_0)t)$$

uh oh...

I transmitted <u>one</u> <u>tone</u> and my FM signal has <u>infinite</u> <u>BW</u>

freq. of signal was mod. → nonlinear op

but $J_n \searrow$ as $n \nearrow$, it decays in freq.

So angle mod. signals are infinite BW

but approximately finite BW

So we can lose a little info in high freq. components and filter before TX and after RX

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$$4x'(t) + x(t-T)$$

$$x(t) \to \boxed{H_1} \to x(t-T)$$

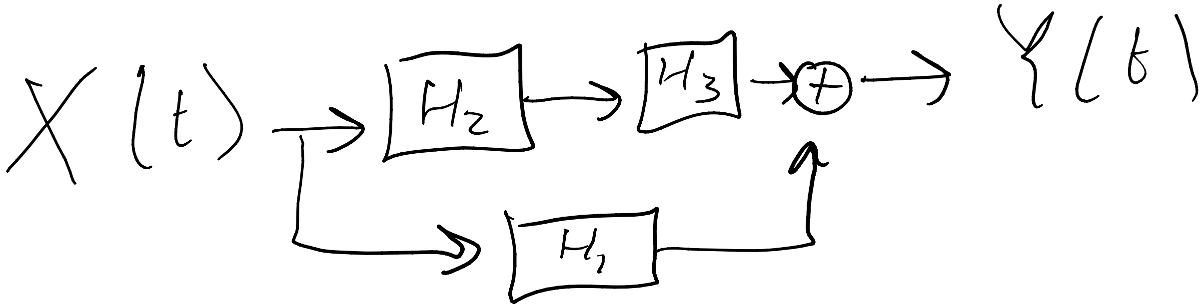
$$H_1 = e^{j\omega T}$$

$$x(t) \to \boxed{H_2} \to 4x(t)$$

$$H_2 = 4$$

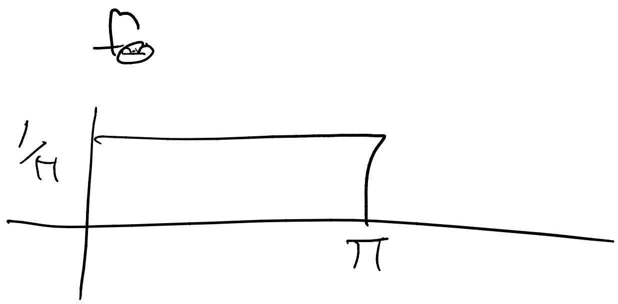
$$x(t) \to \boxed{H_3} \to x'(t)$$

$$H_3 = j\omega$$



$X(t) \rightarrow \boxed{H} \rightarrow Y(t)$

$$H = H_2 H_3 + H_1$$
$$= 4j\omega + e^{j\omega T}$$



$P(\sin\theta \geq 1/2 \ \underline{\text{and}} \ \cos\theta \geq 1/2)$

$P(\sin\theta \geq 1/2)\,P(\cos\theta \geq 1/2)$

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$R_X = E[\sim]$

sample autocorr = sample mean($\sim$)

$\downarrow$ as $N \nearrow$

$E[\sim]$