Review:

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$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{given } P[B] > 0$$

$$\rightarrow P(A \cap B) = P(B) P(A|B)$$

Independent events

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<u>Def.</u> 2 events, A, B are <u>independent</u> iff

$$P(A \cap B) = P(A)P(B)$$

Immediate consequence:

if $P[B] > 0$, A, B are indep, we can write

$$P[A \mid B] = P[A \cap B]/P[B]$$

$$\text{(indep)} \qquad = P[A]\cancel{P[B]}/\cancel{P[B]}$$

$$\to P[A \mid B] = P[A]$$

Knowing B occurred gives us <u>no</u> <u>info</u> about A

Theorems

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<u>THM.</u> If $P[A]=0$, then A is independent of <u>any</u> event B

<u>Proof</u> $P[A]P[B]=0$

and $P[A\cap B]=0$ so A,B are indep.

<u>THM</u> If A,B are indep. then so too are

a) A and B^c

b) A^c and B

c) A^c and B^c

Example

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I roll one red, one blue die

A: I roll a 5 on red

B: the sum of the dice is <u>even</u>

C: The sum of the dice is 10

$P[A] = 1/6$

$P[B] = 1/2$

$P[C] = 3/36$

$$P[A \cap B] = P[A]\,P[B|A] = \frac{1}{6}\,\frac{1}{2} = \frac{1}{12} = P[A]P[B]$$

✓ A, B indep.

$$P[A \cap C] = P[A]\,P[C|A] = \frac{1}{6}\,\frac{1}{6} = \frac{1}{36} \neq P[A]P[C]$$

so A, C are <u>dependent</u>.

One more thing...

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<u>Def.</u> A, B, C are mutually indep. iff

a) A, B indep. B, C indep, A, C indep.

b) $P[A \cap B \cap C] = P[A]P[B]P[C]$

go to n events, with

- all events mutually independent in arbitrary collections

ex: Yahtzee roll 5 indep. dice

note: independence is <u>not</u> mutual exclusivity

A, B <u>m.e.</u> then $P[A \cap B] = 0$

even if $P[A] \neq 0 \neq P[B]$!

most cases ($P[A] \neq 0$, $P[B] \neq 0$)

m.e. $\Rightarrow$ dependence

Total Probability THM

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Theorem Suppose $B_1, \ldots, B_k$ are mutually exclusive and exhaustive (i.e. $\bigcup_{i=1}^{k} B_i = S$) we call $\{B_i\}_{i=1}^{k}$ a partition of S. For any such partition and any event $A \subset S$,

$$\boxed{P[A] = \sum_{i=1}^{k} P[B_i] P[A|B_i]}$$

Proof. $P[B_i]\,P[A|B_i] = P[A \cap B_i]$

$$\rightarrow \sum_i P[B_i]\,P[A|B_i] = \sum_i P[A \cap B_i]$$

$$\text{(m.e.)} \quad = P\left[\bigcup_{i=1}^{k} A \cap B_i\right]$$

$$\text{(exhaustive)} \quad = P[A]$$

$\cup A \cap B_i$

Bayes's Theorem

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$$\boxed{P[B|A] = \frac{P[A|B]\,P[B]}{P[A]}}$$

Can relate one conditional direction to the other

i.e. if I know $P[A|B]$ and the individual probs, I can find $P[B|A]$

More often used formulation:

$\{B_i\}_{i=1}^{k}$ is a partition of S then

$$\boxed{P[B_i|A] = \frac{P[B_i]\,P[A|B_i]}{\sum_{j=1}^{k} P[B_j]\,P[A|B_j]}}$$

TPT to rewrite $P[A]$

Proof. $P[B|A] = \frac{P[A\cap B]}{P[A]}$, $P[A|B] = \frac{P[A\cap B]}{P[B]}$

$P[A\cap B] = P[B]\,P[A|B]$

$P[B|A] = \frac{P[B]\,P[A|B]}{}$

$$P[B/A] = \frac{P[B]\,P[A|B]}{P[A]}$$

Example

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In NY ~1% of people have COVID-19

One common test has a false positive rate of 5%

and a false negative rate of 2%

Q: given I test positive, what is the probability that I have COVID-19?

event V: I have the virus

T^+: I test positive

T^-: I test negative

Want: $P[V|T^+]$

false negative: $\cancel{P[V|T^-]}$ $P[T^-|V]=0.02$

false positive: $P[T^+|V^c]$

$$P[V|T^+]=\frac{P[T^+|V]P[V]}{P[T^+]}$$

$P[V]=0.01$ (given)

$P[T^+|V]=1-P[T^-|V]$ (as T^+ and T^- are exh.)

$=0.98$

$P[T^+] = ?$ $\underline{TPT}$

$$P[T^+] = \underbrace{P[V]}_{.01} \underbrace{P[T^+ | V]}_{0.98} + \overbrace{P[V^c]}^{0.99} \overbrace{P[T^+ | V^c]}^{\text{false}+ \; 0.05}$$

$$= 0.0098 + 0.0495$$

$$P[V | T^+] = \frac{P[T^+ | V] P[V]}{P[V] P[T^+ | V] + P[V^c] P[T^+ | V^c]}$$

$$= \frac{(0.98)(0.01)}{(0.98)(0.01) + (0.99)(0.05)}$$

$$\approx 16.5\%$$

Why!? only 1% of people have it!

Terminology

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$$P[B_k|A] = \frac{P[A|B_k]P[B_k]}{\sum_i P[A|B_i]P[B_i]}$$

Call $P[B_i]$ the "priors"

Call $P[B_k|A]$ the "posterior"

Chapter 2

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Discrete Distributions

$\mathcal{S}$ - sample space. A set of items that may not be numbers ex. {heads, tails}, $\mathbb{C}$

random variables: $X : \mathcal{S} \to \mathbb{R}$

$$\text{range}(X) = \{x \mid x = X(s) \text{ for some } s \in \mathcal{S}\}$$

call this "the space of R.V. X"

usually denote as $S \subset \mathbb{R}$ easier to manipulate than $\mathcal{S}$

For any experiment you can choose ∞ random variables

$X(\text{heads}) = 1$, $Y(\text{heads}) = -53.1$ ← less natural, but valid!

$X(\text{tails}) = 0$, $Y(\text{tails}) = \frac{e}{12\pi^2}$

If $\mathcal{S} \subset \mathbb{R}$, a natural R.V is the identity map, so $S = \mathcal{S}$

and $X(s) = s$

Ex. Die roll $S = \{1, 2, 3, 4, 5, 6\}$

Ex. Die roll $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$

$X(s) = s$, $S = \mathcal{S}$

Probability

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In chapter 1, we covered probabilities on $\mathcal{S}$. We are interested in working in S. Map probability to this new space.

$$\boxed{A \subset S \text{, we define } P[A] \equiv P[X^{-1}(A)]}$$

$$X^{-1}(A) = \{s \in \mathcal{S} \mid X(s) \in A\} \subset \mathcal{S}$$

Ex. I draw a ball from a pot labeled a, b or c

$X(a) = 0$, $X(b) = 1$, $X(c) = 2$

$\mathcal{S} = \{a, b, c\}$, $S = \{0, 1, 2\}$

prob. that I draw an a or b can be written in S as $P[\{0,1\}] = P[\{a,b\}]$

$$X^{-1}(\{0,1\}) = \{a, b\} \subset \mathcal{S}$$

Ex.

Roll a die, $X(1) = X(3) = X(5) = 0$

$X(2) = X(4) = X(6) = 1$

$$P[X=1] = P[\underbrace{\{1\}}_{\subseteq S}] = P[X^{-1}(1)]$$
$$= P[\overbrace{\{2,4,6\}}^{\subseteq S}]$$
$$= 1/2$$

Ex I roll 2 indep. dice, sample space

$$\mathcal{S} = \{(1,1),(1,2),(1,3),\ldots,(6,6)\} \subset \mathbb{R}^2$$

$P[s \in \mathcal{S}] = 1/36$ for any singletons

$X((a,b)) = a+b \in \mathbb{R}$ X maps tuple to the sum of dice

$$S = \{2,3,\ldots,12\}$$

$$P[X=5] = P[X^{-1}(5)] = P[\{(1,4),(4,1),(2,3),(3,2)\}]$$
$$= 4/36$$

$$X^{-1}(5) = \{s \in \mathcal{S} \mid X(s) = 5\} = \{(1,4),(4,1),(2,3),(3,2)\}$$

PMF

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Def. The probability mass function (PMF) of a discrete r.v. X is a function $f: S \to \mathbb{R}$ s.t.

a) $f(x) > 0 \quad \forall x \in S$

b) $\sum_{x \in S} f(x) = 1$

c) $P[X \in A] = \sum_{x \in A} f(x)$

We usually say $f(x) = 0$ for $x \notin S$ and call S the "Support" of f. In this case, $f: \mathbb{R} \to \mathbb{R}$ technically

Uniform dist.

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$|S| = N$ ← # elements

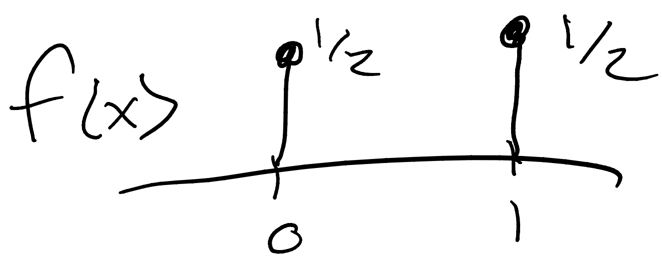
ex. flipping coin $N = 2$

rolling a die $N = 6$

$$f(x) = 1/N \quad \forall x \in S$$

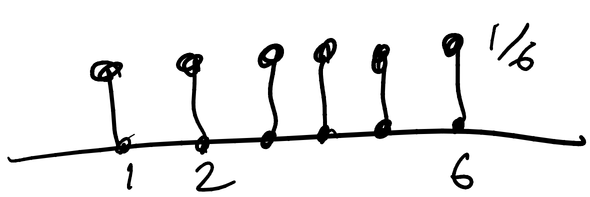
say $X(\text{heads}) = 0$, $X(\text{tails}) = 1$

$f(0) = f(1) = 1/2$



say $Y(1) = 1$, $Y(2) = 2$, $\dots$, $Y(6) = 6$

pmf: $g(y) = 1/6$, $y = 1, 2, \dots, 6$



these are examples of uniform distributions

EX: Not uniform

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X is sum of 2 dice (from before)

$S = \{2, 3, \ldots, 12\}$

$P[X=2] = P[\{(1,1)\}] = 1/36 = P[12=X]$

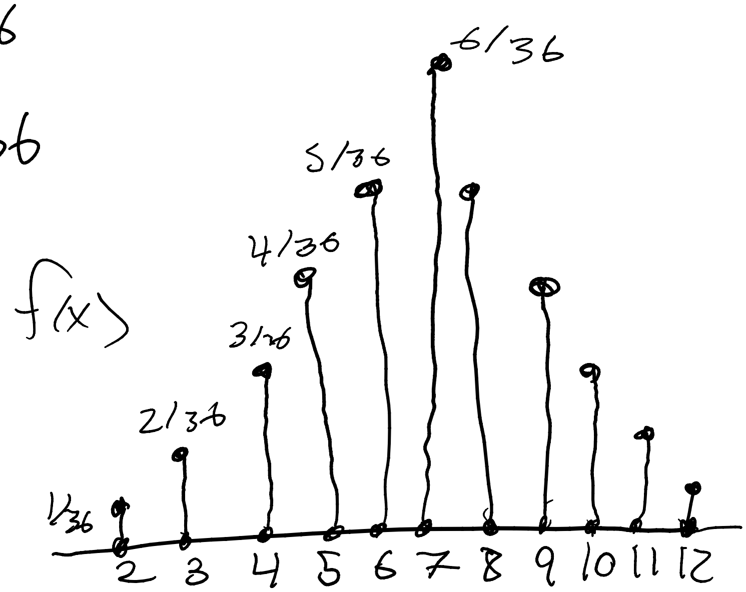
$P[X=3] = P[\{(1,2),(2,1)\}] = 2/36 = P[X=11]$

$P[X=4] = P[X=10] = 3/36$

$P[X=5] = P[X=9] = 4/36$

$P[X=6] = P[X=8] = 5/36$

$P[X=7] = 6/36$



$$f(x) = \begin{cases} 1/36, & x = 2, 12 \\ 2/36, & x = 3, 11 \\ 3/36, & x = 4, 10 \\ 4/36, & x = 5, 9 \\ 5/36, & x = 6, 8 \\ 6/36, & x = 7 \end{cases}$$

$$f(x) = \frac{6 - |7 - x|}{36}, \quad x = 2, \ldots, 12$$

Exam 1

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in 2 weeks - 2/23

- I will post it at 4PM
- we won't have class
- Due at 4PM on 2/24

Covers The material from Lectures 1-3

- all of chapter 1
- chapter 2.1

Focus - methods of enumeration, basic proofs involving laws of probability

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