Enumeration

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$\equiv$ "Counting" ~~sounds easy!~~

understand
- ~~find~~ the underlying distribution (relative frequencies)

distribution is clear in simple cases

EX. Coin flip $\Rightarrow$ Heads has prob 1/2
tails " "

die roll $\Rightarrow$ 1/6 for all sides

more complex: I roll two dice and observe the sum

2, 3, 4, ..., 12

we know 2 less likely than 7.

<u>not</u> uniform

Have a experiment w/ an intuitive distribution

and consider more complex experiments as

a counting problem on possible outcomes of the simpler experiment.

Mult. Principle

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Multiplication Principle

r.e. 1 with n_1 outcomes

r.e. 2 with n_2 outcomes

then the tuple r.e. (r.e. 1, r.e. 2) has $n_1 n_2$ outcomes

ex. r.e. 1 = I flip a coin $n_1 = 2$

r.e. 2 = I roll a die $n_2 = 6$

(r.e. 1, r.e. 2) has outcomes like (Heads, 1)

(Tails, 6)

⋮

there are $2 \cdot 6 = 12$ outcomes

Permutations

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I have n objects, how many distinct ways can I order them?

Ex. $\{A, B, C\}$ are my objects

orders: ABC, BAC, BCA, ACB, CAB, CBA

6 permutations

question: how many permutations exist for a set of n objects?

$$\underset{1}{\underline{\quad\quad}} \quad \underset{2}{\underline{\quad\quad}} \quad \underset{3}{\underline{\quad\quad}} \quad \underset{4}{\underline{\quad\quad}} \quad \cdots \quad \underset{n}{\underline{\quad\quad}}$$

each space can contain only one object
all spaces will be filled
each object appears <u>exactly</u> once

For space 1: I can choose any of my n objects

For space 2: I can choose any object other than the one in space 1, i.e. from $n-1$ objects

3: can choose from $n-2$ objects

$\vdots$

n: "choose" from 1 object

mult. principle: There are $\boxed{n!}$ permutations of n objs

$$n! \equiv n(n-1)(n-2)\ldots 2\cdot 1 \quad \text{for } n \in \mathbb{Z}^+$$

Perm. Exs.

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a) How many distinct words can be created using letters a, b, c, d exactly once?

4 objs, $4! = 4\cdot3\cdot2\cdot1 = \boxed{24}$

b) How many 4-letter words can be created using letters a, b, c, d?

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$
1 2 3 4

in each slot, can choose from 4 objects

$4\cdot4\cdot4\cdot4 = 4^4$

mult. principle

general code

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There are n positions, m objects

$\underset{1}{\_\_} \ \underset{2}{\_\_} \ \underset{3}{\_\_} \ \dots \ \underset{n}{\_\_}$ ← m choices in each pos.

m^n ways to pick a distinct ordered list

PIN: $\underset{1}{\_\_} \ \underset{2}{\_\_} \ \underset{3}{\_\_} \ \underset{4}{\_\_}$ choose from 10 digits
→ 10^4 possible pins

Replacement

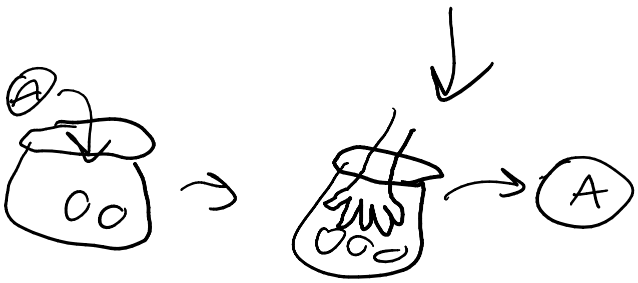
distinction between being able to
"use objects more than once" ≡ with replacement
e.g. PIN code

or "use each object only once" ≡ arranging plants in a display

$\_\_ \ \_\_ \ \_\_$

w/ replacement

A ___ ___ exp. 1



A A ___ exp 2

⋮

W/o replacement

A ___ ___ exp. 1

A C ___ exp 2

⋮

A C B exp3

${}_nP_r$

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$\underset{1}{\underline{\quad}}\ \underset{2}{\underline{\quad}}\ \underset{3}{\underline{\quad}}\ \underset{4}{\underline{\quad}}\ \underset{5}{\underline{\quad}}\ \cdots\ \underset{r}{\underline{\quad}}$

n objects

With replacement: n^r ordered lists

Without replacement: $(r=n)$ $n! = r!$ ordered lists

$(r>n)$ can't say!

$(r<n)$ $n(n-1)(n-2)\cdots(n-(r-1))$ ordered lists

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}} \equiv \#\text{ permutations of } n \text{ objects taken } r \text{ at a time}$$

Note: $0! \equiv 1$, so ${}_nP_n = n!$

Order

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arrange objects in a "list" where order does not matter

$\overline{1}\ \overline{2}\ \cdots\ \overline{r}$, select from n objects w/o replacement

but order doesn't matter

"Combination": $\underline{A}\,\underline{B} = \underline{B}\,\underline{A} \neq \underline{C}\,\underline{B} = \underline{B}\,\underline{C}$

AB, BA, CB, BC counts as only 2 possibilities

For a single combination of n objects taken r at a time, this is an unordered set of r items

$$\{a_1, a_2, \ldots, a_r\}$$

It is true that set can be ordered in $r!$ ways

that means there exist $r!$ ordered lists corresponding to each unordered set (or combination)

$\Rightarrow$ # ... $= \dfrac{{}_nP_r}{\ }$

$\rightarrow$ # Combinations of n items taken r at a time $= \dfrac{{}_nP_r}{r!}$

$$= \boxed{\frac{n!}{(n-r)!\,r!} = {}_nC_r}$$

$_nC_r$

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$$\boxed{{}_nC_r = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}}$$

"n choose r"

# of combinations of n obj's taken r at a time

Ex. How many hands of 5 cards in a deck of 52 cards

$\to \binom{52}{5}$

relates to the "binomial theorem"

$(a+b)^n = (a+b)(a+b)(a+b)\cdots(a+b)$

multiplying out gives many terms of the form

$a^k b^j, \; k+j=n$

how many times does each such term appear?

$a^r b^{n-r}$, how many ways can I select r "a terms" from n "binomial terms"

$\to \binom{n}{r}$ such ways!

Binomial THM

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

Classes

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r objects of type A, $n-r$ objects of type B

how many distinct ways can I order them in n positions?

$\underline{A}$ $\underline{A}$ $\underline{B}$

$\downarrow$

AAB, ABA, BAA 3 possible orderings

distinct from $_nP_r$ problem where all objs are distinguishable

reframe: How do I pick r #s from set $\{1, 2, \ldots, n\}$

why? interpretation is I need r <u>places</u> for objects of class A, and it doesn't matter what order I pick these places in — I just need r positions

once I pick r places for A objects, the <u>rest</u> are B objects

$\underline{A}$ $\underline{A}$ $\underline{B}$, $\begin{matrix} n=3 \\ r=2 \end{matrix}$ $\{1, 2, 3\}$ pick 2 elements

$\underline{A}\ \underline{A}\ \underline{B}$, $r=2$ $\{1,2,3\}$ pick 2 elements

$$\left\{\begin{array}{ll} \underline{A}\ \underline{A}\ \underline{B} & \{1,2\} \\ \underline{A}\ \underline{B}\ \underline{A} & \{1,3\} \\ \underline{B}\ \underline{A}\ \underline{A} & \{2,3\} \end{array}\right.$$

answering: pick r from $\{1,2,\ldots n\}$ unordered

$\binom{n}{r}$ distinct orderings of n objects where r are from 1 class and $n-r$ are from another

$$\rightarrow \boxed{\binom{n}{n-r} = \binom{n}{r}} \qquad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-(n-r))!\,(n-r)!} \leftarrow \binom{n}{n-r}$$

$$= \frac{n!}{r!\,(n-r)!} \leftarrow \binom{n}{r}$$

multinomial

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n positions, K classes of objects

$n_1, n_2, \ldots, n_K$ elements in each class

distinct ordered lists: $\boxed{\dfrac{n!}{n_1!\,n_2!\cdots n_K!} \equiv \dbinom{n}{n_1, n_2, \ldots, n_K}}$

w/replacement, no order

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	replacement	no repl
order	n^r	${}_nP_r$
no order	?	${}_nC_r$

"stars and bars method"

roll a die five times, order doesn't matter

$r=5$, draw 5 stars

$$* \quad * \quad * \quad * \quad *$$

we designate "classes" as values in the sample space $\{1, 2, \ldots, 6\}$ by bars

2 was rolled twice

$$*\,|\; * \;\; * \,|\,|\, * \,|\, * \,|$$

5 was rolled once

6 was rolled 0 times

4 was rolled once

3 was rolled 0 times

1 was rolled once

3 was rolled 0 times

1 was rolled once

$n-1$ bars ($n = \#$ elements in sample space)

bars and stars can go anywhere $(n-1)+r$ positions

r stars

$\#$ arrangements of n items taken r at a time w/ replacement where order doesn't matter is

$$\boxed{\binom{(n-1)+r}{r}}$$

Conditional Prob.

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If I have information about an event, how does this affect the probability?

<u>Ex.</u> probability I roll a 2: $1/6$

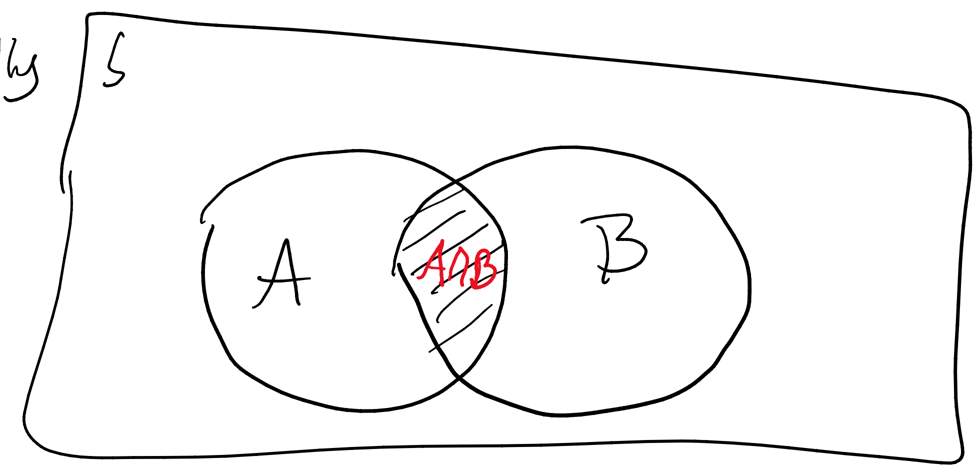
probability I roll a 2 given I roll an even #: $1/3$

<u>Def.</u> The prob. of event A <u>given</u> event B has occurred is

$$\boxed{P[A|B] = \frac{P(A\cap B)}{P(B)}}$$

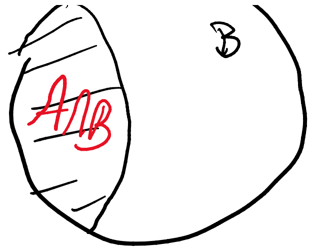
(so long as $P[B] > 0$)

graphically



"given B" our <u>new</u> sample space is B

$P[B|B]$ should be 1



$P[B|B]$ be 1

to find cond. prob, normalize probs s.t. the prob. of $B|B = 1$

$\rightarrow$ divide by $P[B]$

if A occurs <u>given</u> B, then $A \cap B$ occurs

$$\rightarrow P[A|B] = \frac{P[A \cap B]}{P[B]}$$

makes sense graphically

roll a die, A = roll a 2

B = roll an even #

$P[A \cap B] = 1/6$

$P[B] = 1/2$

$$\rightarrow P[A|B] = \frac{1/6}{1/2} = 1/3$$

as we guessed before

Prob. Function

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Note: $P[A|B]$ where $P[B] \neq 0$ satisfies the def. of a probability fncn where $S = B$

a) $P[A|B] \geq 0$ $\forall$ events $A \subset B$ (clear as $\frac{P[A \cap B] \geq 0}{P[B] > 0}$)

b) $P[B|B] = 1$

c) $A_1, \dots, A_k$ mutually exc.

$$P[\bigcup_i A_i | B] = \frac{P[\bigcup_i A_i \cap B]}{P[B]} = \frac{P[\bigcup_i (A_i \cap B)]}{P[B]}$$

$$= \frac{\sum_i P[A_i \cap B]}{P[B]} = \sum_i P[A_i | B] \checkmark$$

by $P[X]$ is a prob. fncn

<u>note:</u> $P[A|B] = P[A \cap B] / P[B]$

$\rightarrow P[A \cap B] = P[B] \, P[A|B]$

Sometimes $P[A|B]$ is clear from intuition

$\rightarrow P[A \cap B]$ which may be less clear

Exs

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I roll 2 dice in order, prob. that the first die lands on 3 and the sum of the dice is 8.

A = roll 3 on the first die

B = sum of dice is 8

want: $P[A \cap B]$ and $= \cap$

as was said, $P[B|A]$ is really clear:
given I roll a 3, sum is 8 iff a 5 is rolled 1/6 chance

$$P[B|A] = 1/6$$

$$P[A \cap B] = P[B|A]\,P[A], \quad P[A] = 1/6$$

$$P[A \cap B] = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$ (jives w/ intuition)

Ex. I draw cards from a deck

what is prob. I draw the 4^{th} king on the 10^{th} draw?

A = draw 3 kings in the first 9 draws

B = draw a king on draw 10

want: $P[A \cap B] = ?$

$P[B|A]$: given I drew 9 cards there are 43 cards left
given I drew 3 kings there is 1 king left

$P[B|A] = 1/43$

$$P[A \cap B] = P[B|A] \, P[A]$$

$P[A] = ?$

1. How many 9-card hands exist?

Order doesnt matter, w/o replacement $\rightarrow \binom{52}{9}$

all 9-card hands are equally likely

$$P[A] = \frac{\text{\# 9-card hands w/ Precisely 3 kings}}{\binom{52}{9}}$$

$$\text{\# hands w/ 9 cards, 3 kings} = \binom{48}{6} \cdot \binom{4}{3}$$

pick 6 non-kings, pick 3 kings

$$P[A \cap B] = P[A]\, P[B \mid A]$$

$$= \boxed{\frac{\binom{48}{6}\binom{4}{3}}{\binom{52}{9}\, 43}}$$

Final Result

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$$P[A\cap B\cap C] = P[A\cap (B\cap C)]$$

$$= P[B\cap C]\,P[A\,|\,(B\cap C)]$$

$$= P[C]\,P[B\,|\,C]\,P[A\,|\,B\cap C]$$

and as the first expression $P[A\cap B\cap C]$ is <u>invariant</u> under exchanging A, B and C

i.e. $P[A\cap B\cap C] = P[B\cap C\cap A] = \ldots$

we can also write

$$P[A\cap B\cap C] = P[A]\,P[B\,|\,A]\,P[C\,|\,A\cap B]$$

$$\vdots$$

useful <u>rarely</u> but often enough in computing probs of a triple intersection