Ma-224-E

Tuesday, January 26, 2021 3:50 PM

Goal: create/develop a mathematical framework for <u>uncertainty</u>

Applications

- Statistical/random model
- Noise
- Statistical mechanics/thermodynamics

Random Experiment

Tuesday, January 26, 2021 4:18 PM

~ any process w/ a well-defined output that <u>you</u> can't predict

Outcome need not be a number

<u>Ex.</u> Flipping a coin $\rightarrow$ {Heads, tails}

Rolling a die $\rightarrow \{1, \ldots, 6\}$

Sum of rolling 1000 dice $\rightarrow \{1000, \ldots, 6000\}$

Reading T off thermometer $\xrightarrow{K} (0, \infty)$

output might live on a continuum

<u>Def.</u> The set of possible outputs for a rand. exp. is called the <u>Sample Space</u> (S) or <u>Output Space</u>

Sample space can be: Discrete ← "countable" ≈ "like" $\mathbb{Z}$ — "finite set"

Continuous ← "uncountable" ~ "like" $\mathbb{R}$

"S" may not contain #s

Random Variable

Tuesday, January 26, 2021 4:30 PM

<u>Def</u> a <u>random variable</u> (usually X, Y or Z) associated w/ a rand. exp. w/ sample space S is a <u>function</u> from $S \to \mathbb{R}$

<u>Ex.</u> Coinflip $X : \{\text{Heads}, \text{Tails}\} \to \mathbb{R}$ ~~not~~ onto

note that there are infinitely many R.V.s associated w/ each exp!

$$\begin{cases} X(\text{Heads}) = 0 \\ X(\text{Tails}) = 1 \end{cases}$$ one possible RV

$$Y : \{\text{Heads}, \text{Tails}\} \to \mathbb{R}$$

$$\begin{cases} Y(\text{Tails}) = 0 \\ Y(\text{Heads}) = 1 \end{cases}$$

$$\begin{cases} Z(\text{Tails}) = 6.02 \times 10^{23} \\ Z(\text{Heads}) = -\pi^2/12 \end{cases}$$ valid too!

RVs (Cont.)

Tuesday, January 26, 2021 4:37 PM

RVs also either "discrete" or "cts"

Ex. Temperature in Kelvin $S = (0, \infty)$

$X : (0,\infty) \to \mathbb{R}$

one r.v. is $X(T) = T \quad \forall T \in S$

another is $Y(T) = \lfloor T \rfloor \quad \forall T \in S$

$\text{range}(X) = (0, \infty)$ <u>cts</u>

$\text{range}(Y) = \mathbb{N}$ <u>discrete</u>

Call a r.v. w/ cts <u>output</u> a cts r.v.

" " discrete <u>output</u> a discrete r.v.

<u>Can</u> have cts $S \to$ discrete r.v.

if S is <u>discrete</u> r.v. <u>must</u> be discrete

ex. $\{\text{heads}, \text{tails}\} \to$ <u>at most</u> maps to 2-value set maps to 2-value set

"at most 2" $Z(\text{heads}) = Z(\text{tails}) = 0$

valid r.v.

r.v. not necessarily injective

Distributions

Tuesday, January 26, 2021 4:43 PM

Rand. exps have underlying <u>distributions</u>

→ outputs have relative frequencies of occurrence

Ex. flipping a coin heads/tails each happen about ½ of the time

(Discrete case) If I repeat an experiment N times and record outputs

$S = \{s_1, s_2, \dots\}$ and each occurs

$n_1, n_2, \dots$ times in the N experiments

as $N \nearrow$, values of $\frac{n_i}{N}$ <u>stabilize</u> at the <u>underlying relative frequencies</u>

we call $p_i = \lim_{N \to \infty} \frac{n_i}{N}$ the probability of s_i

in S, the sample space

PMF

Tuesday, January 26, 2021 4:50 PM

For a discrete r.v. X, $\text{range}(X) = \{x_1, x_2, \dots\}$ we define the probability mass function of X (pmf) as

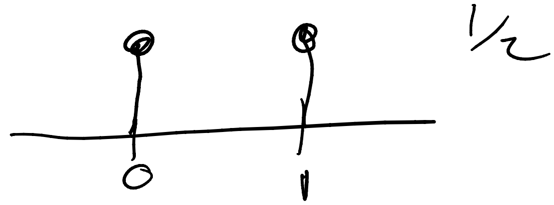
$$f(x_i) = P(x_i) \leftarrow ?$$

$$= P(X^{-1}(x_i)) \leftarrow \text{in } S$$

Ex. $X(\text{Heads}) = 1$, $X(\text{Tails}) = 0$

$$f(0) = P(X^{-1}(0)) = P(\text{tails}) = 1/2$$

$$f(1) = 1/2$$

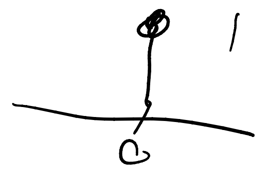


pmf is defined on range(X) because then the pmf is a fnctn of real values

→ is a fnctn we understand

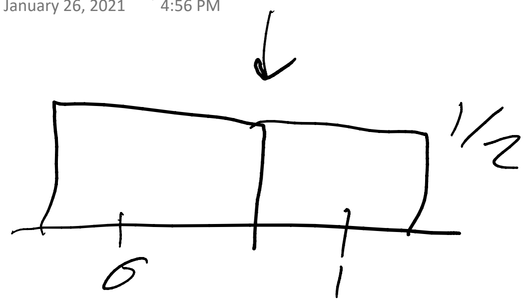
Ex. $Z(\text{heads}) = Z(\text{tails}) = 0$

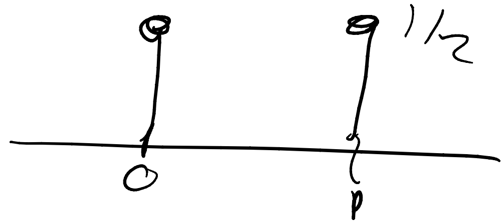
$$\to f(0) = P(Z^{-1}(0)) = P(\{heads, tails\})$$
$$= 1$$



Probability histogram

Tuesday, January 26, 2021 4:56 PM





for $f(x)$

if you measure from an exp. N times, can also construct a histogram

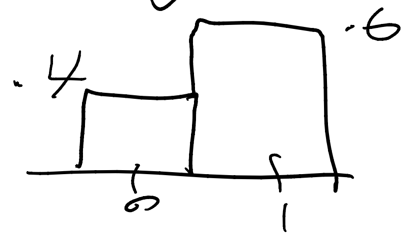
<u>ex.</u> 5 coinflips return $\{H, H, H, T, T\}$

$X(H)=1,\ X(T)=0,\ P(H)=3/5=.6$

$P(T)=2/5=.4$

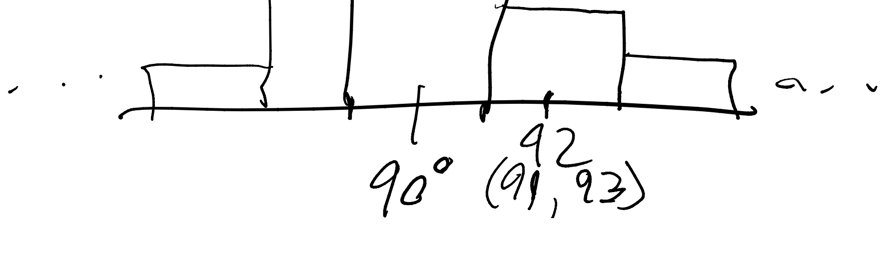
$f(0)=p(T)=.4$

$f(1)=p(H)=.6$



for cts r.v.s, construct a density hist.

← $P(T\in[89,91))$ measured exp.



"binning"

1 92
90° (91, 93)
(89, 91)

↳ book mentions "Simpson's Paradox"
read about it!

Set Theory review

Tuesday, January 26, 2021 5:01 PM

$A \cup B$, $A = \{1, 2\}$, $B = \{2, 3\}$

or

$A \cup B = \{1, 2, 3\}$

$A \cap B = \{2\}$

and

not in B

$A - B = A \cap B^c = \{1\}$

$\emptyset$ empty set $\{\}$

$x \in A^c$ if $x \notin A$

ill-defined

$A = \{1, 2\}$

$x = 3 \notin A$

$x = 3.4 + 2i \notin A$

$x = (1, 2) \notin A$

$x =$ "Heads" $\notin A$

"$\notin$" defined w.r.t. some "universal set"

$A \subset \mathbb{N}$, $B \subset \mathbb{N}$, $\mathbb{N}$ universal set

$A^c = \{a, 3, 4, 5, \ldots\}$ well-defined

$U \equiv$ universal set

$U^c = \emptyset, \ \emptyset^c = U$

Events

Tuesday, January 26, 2021 5:11 PM

Universal set $= S$ (sample space)

Def an *event* is any subset of S

Ex. roll a die $S = \{1, 2, 3, 4, 5, 6\}$

some events: $\{1\} = A$ "event I roll a 1"

a set

$B = \{1, 3, 5\}$ "roll an odd"

$C = \{2, 3, 5\}$ "roll a prime"

S itself is "an event"

Def events A, B are "mutually exclusive" (m.e.)

if $A \cap B = \emptyset$

more generally, $A_1, A_2, \ldots, A_k$ are m.e. if

$A_i \cap A_j = \emptyset$ for all i, j

no 2 sets have *any* elements in common

distinct from $A_1 \cap A_2 \cap \ldots \cap A_k = \emptyset$

ex. $\{1,2\} \cap \{2,3\} \cap \{4,5\} = \emptyset$

<u>but</u> not m.e. as $\{1,2\} \cap \{2,3\} \neq \emptyset$

<u>Def.</u> events $A_1, \ldots, A_k$ are <u>exhaustive</u>

if $A_1 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i = S$, the universal

ex. temp. has $S = (0, \infty)$

$A = (0, 273)$, $B = [273, \infty)$, A, B are exhaustive

$C = (273, \infty)$, A, C are <u>not</u> exhaustive

$A \cup C \not\ni 273 \in S$

A, B, C are together exhaustive

Probability

Tuesday, January 26, 2021 5:21 PM

Def. Probability is a function defined on events in S (defined on subsets of S, i.e. $\mathcal{P}(S)$) w/ real values satisfying:

1) $P[A] \geq 0 \quad \forall A \subset S$

2) $P[S] = 1$

3) if $A_1, A_2 \ldots$ (countable) are m.e.

then $P[\bigcup_i A_i] = \sum_i P[A_i]$

Ex. die roll A = I roll a 2 $= \{2\}$

B = I roll an odd number $= \{1,3,5\}$

A, B m.e.

$P[A] = 1/6$, $P[B] = 1/2$

$P[A \cup B] = \frac{1}{6} + 1/2 = 2/3 \qquad A \cup B = \{1,2,3,5\}$

note: we didn't "derive" the probabilities but

Note: We didn't derive the probabilities but rather intuited them from personal experience

Jone THMs

Tuesday, January 26, 2021 5:31 PM

THM 1.2-1 for any event A, $P[A^c] = 1 - P[A]$

Proof. $A \cap A^c = \phi$, so by def of Prob. part (3)

we have $P[A] + P[A^c] = P[A \cup A^c]$ as A, A^c are m.e.

now $A \cup A^c = S$, and $P[S] = 1$ by def of prob. (2)

so $P[A^c] = 1 - P[A]$ //

THM 1.2-2 $P[\emptyset] = 0$

Proof. by THM 1.2-1, $P[S^c] = 1 - P[S]$

(by def. of prob (2)) $= 1 - 1$

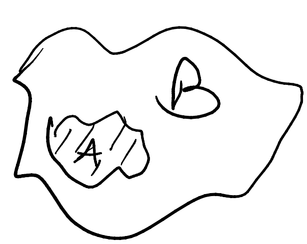
$= 0$

$S^c = \emptyset$. //

THM 1.2-3 If $A \subset B$, $P[A] \leq P[B]$

Proof. $B = A \cup (B - A)$

$= A \cup (B \cap A^c)$



$A, B \cap A^c$ m.e.

by prob. def. (3), $P[B] = P[A] + P[A^c \cap B]$

by def. (1), $P[A^c \cap B] \geq 0$, $\geq P[A]$

THM 1.2-4 $P[A] \leq 1$ For all A event

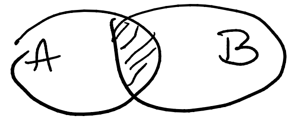
Proof. From 1.2-3, $P[A] \leq P[B]$ if $B \supset A$

thus, any event $A \subset S$, so $P[A] \leq P[S] = 1$ (def (2))

for any $A \subset S$

THM 1.2-5 $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Proof.



$A \cup B = A \cup (B - A) = A \cup (B \cap A^c)$

$A, B \cap A^c$ m.e.

$A \cup B = A \cup (B - A) = A \cup (B \cap A^c)$

$A, B \cap A^c$ m.e.

by def. (3), $P[A \cup B] = P[A] + P[B \cap A^c]$ $\circledast$

Can also write $B = (A \cap B) \cup (B - A)$

$= (A \cap B) \cup P[B \cap A^c]$

$A \cap B, B \cap A^c$ m.e.

by def. (3), $P[B] = P[A \cap B] + P[B \cap A^c]$

rearranging $P[B \cap A^c] = P[B] - P[A \cap B]$

plugging into $\circledast$, $P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Final Example

Tuesday, January 26, 2021 5:46 PM

I have 52 cards, what is the prob. I draw a king or a diamond

$A = \{\text{I draw a king}\}$, $P[A] = 4/52$

$B = \{\text{I draw a diamond}\}$, $P[B] = 13/52$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \cap B] = P[\text{"I draw the king of diamonds"}] = 1/52$$

$$P[A \cup B] = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \boxed{\frac{16}{52}}$$